# Integrable systems on $\mathbb{S}^3$

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## 1. Introduction and objectives

The aim of this poster is to present

- a generalization of the conclusions of Fomenko and others to non Hamiltonian systems defined on $\mathbb{S}^3$.
- an invariant for this kind of flow.
- the study of the periodic orbits, as done by Wada.
- obstructions of integrability from the type of knot of the periodic orbits in the systems studied here.

## 2. Knots and Morse-Bott functions

Let $M$ be a manifold.

- A **knot in** $M$ is the image of an immersion $h : S^1 \to M$. Two knots are equivalent if there exists a preserving orientation homeomorphism $H : M \to M$ that conjugates the immersions of the knots, i. e. $H(h_1(S^1)) = h_2(S^1)$. A **link** is a set of non intersecting knots.
- A function $f : M \to \mathbb{R}$ is called **Morse-Bott function** (MB function from now on) if its singular points are organized as non degenerate smooth critical or singular manifolds. Here a critical manifold of $f$ is called degenerate if the Hessian of $f$ is non degenerate on normal planes of this submanifold.
- The level sets of $f$ are $I_a(f) = \{p \in M : f(p) = a\}$.
- If $p$ is a critical point then $f(p) = a$ is called a **critical value** and the corresponding level set is called a singular level.
- The level sets of a MB function on $M$ define a **singular foliation** $\mathcal{F}(\mathcal{M})$ **on** $M$.
- If $dimM = n$, each leaf that has codimension less than $n-1$ or it is not a submanifold of $M$ is called a singularity of the foliation $\mathcal{F}(\mathcal{M})$.
- Let $a$ be a critical value of $f$. The set of critical points of $f$ contained in $I_a(f)$ is denoted by $S_a(f)$. Moreover, the singular set of a MB function $f$ does not need to coincide with the singularities of the foliation.
- Denote by $Sing(\mathcal{F}(S^3))$ the set of singularities of $\mathcal{F}(S^3)$. They are classified as saddle singularities $Sad(\mathcal{F})$ and center singularities $Cen(\mathcal{F})$ according to the index of the normal planes.

## 3. Integrable vector fields

Denote by

- $\psi_{MB}(M)$ the set of all $C^r$-vector field $v$ that admits a MB function $f$ as a first integral ($r \geq 2$).
- $\psi_{NMB}(M^n)$ the set of all non singular vector fields in $\psi_{MB}(M)$ without saddle connections.

## 4. Level sets and handles to non singular vector fields

Let $(v, f)$ be a pair where $v$ is a vector field and $f$ a particular first integral of $v$.

**Proposition 1.** *Let $(v, f)$ be a pair where $v \in \psi_{NMB}(M)$ and $a$ a regular value of $f$. Then each connected component of $I_a(f)$ is homeomorphic to a 2-dimensional torus.*

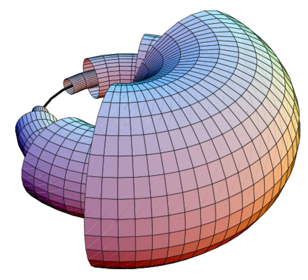
If $a$ is a critical value then

**Proposition 2.** *Let $f$ be a MB first integral for a vector field $v \in \mathbb{S}^3$, then $S_a(f)$ is union of circles or torus and it is an invariant set of $v$.*

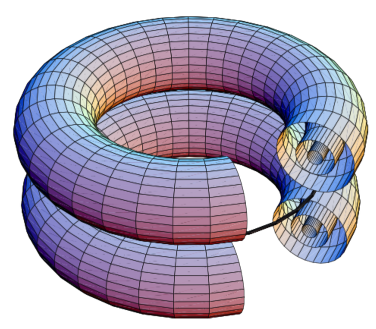
Let $s$ be a critical connected manifold of $S_a(f)$, $H_a(f, s)$ denotes a connected component of $I_{[a-\epsilon, a+\epsilon]}(f)$.

We can think $H_a(f, s)$ as a closed neighborhood of the invariant manifold.

As $f$ is constant on each connected component of the border of $H_a(f, s)$ we can decompose $M$ using these handles. Moreover, when $s$ is a center singularity $H_a(f, s)$ is a closed regular neighborhood of this singularity.



**Definition 1.** *Denote by $\Sigma(f)$ the manifold $M$ minus the level sets of saddle singularities $\sigma$. We call basin of $\sigma$ in $\Sigma(f)$ to each connected component of $\Sigma(f)$.*



In the next theorem $F_p$ denotes a disk with $p \geq 1$ holes and $D^2$, an open unity disk.

**Theorem 1.** *Let $\sigma$ be a saddle periodic orbit contained in $S_a(f)$ without connection by means of a separatrix with another saddle periodic orbit, then $H_a(f, \sigma)$ is homeomorphic to:*

1. *$F_2 \times S^1$, if $H_a(f, \sigma)$ is orientable.*
2. *$D^2 \times S^1 - \Psi$, where $\Psi$ is a regular neighborhood of a $(2, 1)$-cable of $\{0\} \times S^1$, if $H_a(f, \sigma)$ is not orientable.*

## 5. Links of singular periodic orbits

As handles are attached in an essential way and the topological characterization is the same that in the Hamiltonian system we conclude the following

**Proposition 3.** *The link of singular periodic orbits of a non singular MB integrable vector field on $\mathbb{S}^3$ without saddle connections is made by applying operations IV, V and VI to a Hopf link.*

These are the operations defined by Wada in [5]
Let $\mathcal{L}(f)$ be this link.

**Proposition 4.** *$\mathcal{L}(f)$ has the following properties:*

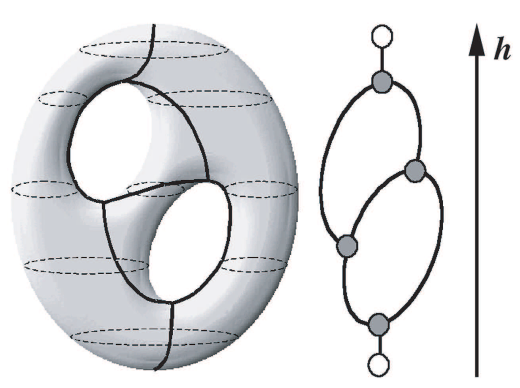
1. *Let $s$ be a periodic orbit of a vector field on $\mathbb{S}^3$ with a first integral of Morse-Bott type, then it is an generalized torus knot.*
2. *It is a not splitting link.*
3. *If the periodic saddle orbits are orientable then $|\sigma^1| + 2 \leqq |\varsigma^1| \leqq |2\sigma^1| + 2$, where $\sigma^1$ and $\varsigma^1$ denote the number of saddle points of dimension 1.*

The proof of this proposition follows similar arguments to those in Fomenko [4].

## 6. Graph link

Let $M$ be a manifold and $f : M \to R$ a continuous function. The space obtained from $M$ by contracting each connected component of the level sets to a point is called **Reeb graph of** $f$, $R_G(f)$.

This graph is said to be a **directed graph** if the value of $f$ at the origin of an edge is lower than the value of the ebd the edge.




**Definition 2.** *The graph link of the pair $(v, f)$, where $v \in \psi_{NMB}(M)$, consists of the set $R_G(f)$, the link $\mathcal{L}(f)$ and a map from $R_G(f)$ to $\mathcal{L}(f)$ sending the vertex corresponding to the critical value $a$ to the periodic orbits on $I_a(f)$.*

## 7. Vector fields completely integrable

**Definition 3.** *We will say that a vector field $v(M^n)$ is completely integrable if it has two independent first integrals $f_1, f_2$. We will denote by $(v, f_1, f_2)$ the vector field and its integrals.*

We will assume:
I. $f_i$ are MB functions.
II. $f_i$ is a MB function on each regular level set $I_a(f_j), j \neq i$.

**Proposition 5.** *The round handle decomposition of $S^3$ defined by $f_1$ is topologically equivalent to the round handle decomposition defined by $f_2$.*

**Definition 4.**

- *$\mathcal{L}(f_1, f_2)$ consists of $\mathcal{L}(f)$ and a periodic orbit of the basin of each $\varsigma^1$.*
- *The graph link of $(v, f_1, f_2)$ consists of the set of $R_G(f)$ and $\mathcal{L}(f_1, f_2)$ and a map from $R_G(f)$ to $\mathcal{L}(f_1, f_2)$ sending each vertex to a periodic orbit of $\mathcal{L}(f_i), i = 1$ or $i = 2$ and an edge of $R_G(f)$ to an orbit of $\mathcal{L}(f_1, f_2) - \mathcal{L}(f_i)$.*

The graph link is an invariant of orbital equivalent systems.

### 7.1 Example

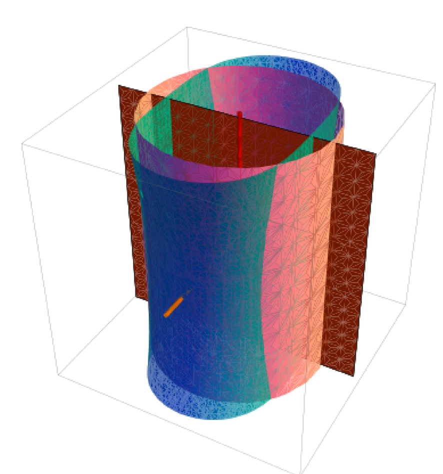
One easy example of MB completely integrable system defined over the sphere $x_1^2 + x_2^2 + x_3^2 + x_4^2 = 1$: :

$$
\begin{aligned}
\frac{dx_1}{dt} &= x_2x_3(3x_2^2 + x_1^2(1 + x_4^2)), \\
\frac{dx_2}{dt} &= -x_1x_3(x_1^2 - x_2^2(x_4^2 - 3)), \\
\frac{dx_3}{dt} &= -x_1x_2x_4^2(2 + x_1^2 + x_2^2 - x_2^4), \\
\frac{dx_4}{dt} &= -x_3x_2x_3x_4(x_4^2 - 2).
\end{aligned} \tag{1}
$$

It has as MB first integrals

$$
\begin{aligned}
f_1 &= x_1^2 + x_2^2(1 + x_4^2), \\
f_2 &= (x_1^2 + x_2^2)(2 - x_4^2).
\end{aligned} \tag{2}
$$

- Each first integrals has the circle $x_1 = 0, x_2 = 0$ as a singular center.
- The singularity of $(f_1, f_2)$ contains also the 2-sphere $x_4 = 0$. This sphere intersects a torus around $\varsigma_1^1$, $(x_1^2 + x_2^2)(2 - x_4)^2 = \epsilon$ in the two parallels $x_3^2 = 1 - \frac{\varepsilon}{2}$. This is where regular tori $I_a(f_1)$ and $I_b(f_2)$ for some values of $a$ and $b$ are tangent.
- There are also periodic orbits as intersection of invariant tori. As $B(\varsigma_1)$ is all $S^3$ minus the center one dimensional circles, the dynamics of the system consists in trivially unknotted periodic orbits around the centers.



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