Chaotic behaviour of linear operators on invariant sets

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Introduction

We study mixing properties (topological mixing and weak mixing of arbitrary order) for nonautonomous linear dynamical systems that are induced by the corresponding dynamics on certain invariant sets.

The type of nonautonomous systems are defined by a sequence $T_\infty = (T_i)_{i\in\mathbb{N}}$ of linear operators $T_i : X \to X$ on a topological vector space X, for all $i \in \mathbb{N}$, such that there is an invariant set Y (i.e, $T_i(Y) \subset Y$, $i \in \mathbb{N}$), for which the restricted dynamics $T_\infty|_Y = (T_i|_Y)_{i\in\mathbb{N}}$ satisfies certain mixing property. We then obtain the corresponding mixing property on the closed linear span of Y.

Then we study measure properties of dynamical systems and we construct strongly mixing invariant measures with full support for operators on F-spaces which satisfy the Frequent Hypercyclicity Criterion.

We will generally work with a separable Banach space X and $T_n : X \to X$ will be a sequence of linear and continuous maps(operators). We are concered with the general dynamics of $(T_n)_n$.

- $(T_n)_n$ is called *topologically transitive* if for any pair of nonempty open sets $U,V \subset X$ there exists an $n \in \mathbb{N}$ such that $T_n(U) \cap V \neq \emptyset$.
- $(T_n)_n$ is called *mixing* if for any pair of nonempty open sets $U,V \subset X$ there exists some $n_0 \in \mathbb{N}$ such that $T_n(U) \cap V \neq \emptyset$ for every integer $n \geq n_0$.
- $(T_n)_n$ is *weakly mixing* of order n if, for any nonempty open sets $U_1, \dots, U_n, V_1, \dots, V_n$ and for any $N > 0$ there is $k > N$ such that $T_k(U_i) \bigcap V_i \neq \emptyset$ for $i = 1, \dots, n$. If $(X, (T_n)_n)$ is weakly mixing of order n for every $n \geq 2$ then we say that it is *weakly mixing* of all orders.
- $(T_n)_n$ is called *hypercyclic* if there is some $x \in X$ whose orbit $Orb(x,(T_n))$ is dense in X. In that case, x is called a *hypercyclic vector* for $(T_n)_n$.
- $(T_n)_n$ is called *chaotic* if it is hypercyclic and the set of perioidc points $Per((T_n))$ is dense in X.

The corresponding notions for a single operator $T : X \to X$ are defined considering the sequence $(T^n)_n$.

- A Borel probability measure μ has *full support* if for all non-empty open set $U \subset X$ $\mu(U) > 0$.
- A measurable map $T : (X, \mathfrak{B}, \mu) \to (X, \mathfrak{B}, \mu)$ is called a *measure preserving* transformation if $\mu(T^{-1}(A)) = \mu(A)$ for all $A \in \mathfrak{B}$.
- T is said to be *strongly mixing* with respect to μ if

$$lim_{n\to\infty}\mu(A \cap (T^{-n})(B)) = \mu(A)\mu(B) \qquad (A, B \in \mathfrak{B})$$

.

1. Proposition

Let $T : X \to X$ be an operator and let K be an absolutely convex T-invariant set such that $T|_K$ is transitive (respectively weakly-mixing, mixing, chaotic), then $T|_{\overline{span(K)}}$ also verifies the same property.

2. Example

Let T be a weighted backward shift on the space ℓ^p given by

$$T(x_1, x_2, x_3, \dots) = (w_2x_2, w_3x_3, w_4x_4, \dots)$$

with $\sum_{n=1}^{\infty} \frac{1}{\prod_{k=1}^{n} |w_k|^p} < \infty$. Let K be the following subset:

$$K = \{ \quad x \in \ell^p \quad ; \quad |x_k| \prod_{j=1}^{k} |w_j| \leq 1, \forall k \geq 1\}$$

with $w_1 = 1$. Then K is absolutely convex, T-invariant and $T|_K$ is chaotic. As $\overline{span(K)} = \ell^p$ from theorem before we deduce that $T : \ell^p \to \ell^p$ is chaotic.

3. Mixing properties induced by invariant sets

Let X be a Banach space and let the system $(X, (T_n)_n)$, where $\{T_n : X \to X \; ; \; n \in \mathbb{N}\}$ is a sequence of operators such that $T_n(Y) \subset Y$ for every $n \in \mathbb{N}$ and for certain $Y \subset X$ with $0 \in Y$. We consider $Z := \overline{span(Y)}$.

(1) If $(Y, (T_n|_Y)_n)$ is weakly mixing of all orders then $(Z, (T_n|_Z)_n)$ is also weakly mixing of all orders.

(2) If $(Y, (T_n|_Y)_n)$ is mixing then $(Z, (T_n|_Z)_n)$ is also mixing.

4. Example

Let $\{p_n : I \to I \; ; \; n \in \mathbb{N}\}$ be a sequence of polynomials on an interval I that contains 0 such that $p_n(0) = 0$, $n \in \mathbb{N}$, and the corresponding generated nonautonomous dynamical system (I, p_∞) is weakly mixing of order 3. It is known that (I, p_∞) is weakly mixing of all orders. We will embed (I, p_∞) in a linear nonautonomous dynamical system (X, T_∞).

To do so we set

$$X = \{(x_i)_i \in \mathbb{C}^{\mathbb{N}} \; ; \; \exists r > 0 \text{ such that } \sup_i |x_i| r^i < \infty\}.$$

Actually, we can identify via monomial expansion $X = \mathcal{H}(0)$, the space of holomorphic germs at 0. That is, X consists of the functions that are (defined and) holomorphic on a neighbourhood of 0. X is endowed with the natural topology as inductive limit.

We define the embedding $\phi : I \to X$ as $\phi(x) = (x, x^2, x^3, \dots)$. Given $n \in \mathbb{N}$, we set the operator $T_n : X \to X$,

$$T_n e_k = \sum_{j=k}^{km_n} \alpha_{k,j} e_j, \quad k \in \mathbb{N},$$

where $m_n = \deg(p_n)$ and $p_n(x)^k = \sum_{j=k}^{km_n} \alpha_{k,j}$. The selection of the sequence space X easily gives that T_n is a (well-defined) operator on X. Also, a simple computation shows that $T_n \circ \phi = \phi \circ p_n$. Let $Y := \phi(I)$. We observe that $span(Y)$ is dense in X by the Hahn-Banach theorem. Indeed, since the dual of X is

$$X' = \{(y_i)_i \in \mathbb{C}^{\mathbb{N}} \; ; \; \sum_{i=1}^{\infty} |y_i| R^i < \infty \text{ for all } R > 0\},$$

which can be identified with the space of entire functions, we have that $\langle \phi(x), (y_i)_i \rangle = \sum_i y_i x^i = 0$ for some $(y_i)_i \in X'$ and for all $x \in I$, implies $y_i = 0$ for every $i \in \mathbb{N}$ because an entire function that is annihilated on a set with accumulation points should be identically 0. The hypothesis of Theorem before are satisfied, and (X, T_∞) is weakly mixing of all orders.

5. Corollary

Let $T = (T_1, \dots, T_n)$ be a commuting tuple of operators defined on a topological vector space X. Let $x \in X$ such that $Orb(x,T) = \{(T_n^{k_n} \circ \dots \circ T_1^{k_1})x \; ; \; k_i \geq 0 \text{ for all } i\}$ is somewhere dense in X. Let $(R_n \circ \dots \circ R_1)_{n\in\mathbb{N}}$ be an arrangement of $\{T_n^{k_n} \circ \dots \circ T_1^{k_1} \; ; \; k_i \geq 0 \text{ for all } i\}$ as a linear nonautonomous discrete system, and let $Y := \overline{Orb(x,T)}$. If $(Y, R_\infty|_Y)$ is weakly mixing of all orders then $Orb(x,T)$ is everywhere dense.

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6. Invariant measures and the frequent hypercyclicity criterion

We recall that a series $\sum_n x_n$ in X *converges unconditionally* if it converges and, for any 0-neighbourhood U in X, there exists some $N \in \mathbb{N}$ such that $\sum_{n\in F} x_n \in U$ for every finite set $F \subset \{N, N+1, N+2, \dots\}$.

Let T be an operator on a separable F-space X. If there is a dense subset X_0 of X and a sequence of maps $S_n : X_0 \to X_0$ such that, for each $x \in X_0$,

- $\sum_{n=0}^{\infty} T^n x$ converges unconditionally
- $\sum_{n=0}^{\infty} S_n x$ converges unconditionally, and
- $T^n S_n x = x$ and $T^m S_n x = S_{n-m} x$ if $n > m$.

then there is a T-invariant strongly mixing Borel probability measure μ on X with full support.

7. Sketch of the proof

The idea behind the proof is to construct a "model" probability space $(Z, \overline{\mu})$ and a (Borel) measurable map $\Phi : Z \to X$, where $Z \subset \mathbb{N}^{\mathbb{Z}}$ is such that $\sigma(Z) = Z$ for the Bernoulli shift $\sigma(\dots, n_{-1}, n_0, n_1, \dots) = (\dots, n_0, n_1, n_2, \dots)$, $\overline{\mu}$ is a σ^{-1}-invariant strongly mixing measure, $Y := \Phi(Z)$ is a T-invariant dense subset of X, $\Phi\sigma^{-1} = T\Phi$.

Then the Borel probability measure μ on X defined by $\mu(A) = \overline{\mu}(\Phi^{-1}(A))$, $A \in \mathfrak{B}(X)$, is T-invariant and strongly mixing.

8. Corollary

Let $T : X \to X$ be a chaotic bilateral weighted backward shift on a sequence F-space X in which $(e_n)_{n\in\mathbb{Z}}$ is an unconditional basis. Then there exists a T-invariant strongly mixing Borel probability measure on X with full support.

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