Global Dynamics in the Poincaré Ball of the Chen System Having Invariant Algebraic Surfaces

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1. Introduction

The Chen system is the three-parameter family of quadratic polynomial differential equations given by

$$\dot{x} = a(y - x), \quad \dot{y} = (c - a)x - xz + cy, \quad \dot{z} = xy - bz, \qquad (1)$$

where $(x, y, z) \in \mathbb{R}^3$ are the state variables and a, b, c are real parameters. The dots denote differentiation with respect to time t.

System (1) was firstly studied in [1]. It exhibits chaotic phenomena for suitable choices of the values of the parameters a, b and c. In this way, some important properties of system (1) are similar to the properties of the well-known Lorenz system

$$\dot{x} = ry - x - yz, \quad \dot{y} = \sigma(x - y), \quad \dot{z} = -bz + xy. \qquad (2)$$

In [2] the authors introduce a unified chaotic system containing the Lorenz and the Chen systems as dual systems at the two extremes of its parameter spectrum. This unified system represents the transition from the Lorenz to the Chen system and is chaotic over the entire range of the parameter.

In this work we performed a global analysis of the dynamics of the Chen system (1). First, we give the complete description of its dynamics on the sphere at infinity. For six sets of the parameter values the system has invariant algebraic surfaces. In these cases we provide its global phase portrait and give a complete description of the α– and ω–limit sets of its orbits in the Poincaré ball, including its boundary $\mathbb{S}^2$, i.e. in the compactification of $\mathbb{R}^3$ with the sphere $\mathbb{S}^2$ of the infinity. Moreover, we prove the existence of a family with infinitely many heteroclinic orbits contained on invariant cylinders when system (1) has a line of singularities and a first integral, which indicates the complicated dynamical behavior of the Chen system solutions even in the absence of chaotic dynamics.

All the results are precisely stated and proved in [5].

2. Main Results

Let $\mathbb{R}[x, y, z]$ be the ring of the real polynomials in the variables x, y and z. We say that $F = F(x, y, z) \in \mathbb{R}[x, y, z]$ is a *Darboux polynomial* of system (1) if it satisfies $(\nabla F) \cdot (P, Q, R) = kF$, where $k = k(x, y, z)$ is a real polynomial of degree at most 1, called the *cofactor* of $F(x, y, z)$, and ∇F denotes the gradient of F. If the cofactor is zero, then $F(x, y, z)$ is a polynomial first integral of system (1). If $F(x, y, z)$ is a Darboux polynomial, then the surface $F(x, y, z) = 0$ is an *invariant algebraic surface*.

The following proposition gives a summary on the invariant algebraic surfaces of system (1), it is due to Lu and Zhang [3].

Proposition 1. *If $a = 0$ then the phase portrait of system (1), restricted to each plane $x =$constant is determined by a linear differential system. If $a \neq 0$ then system (1) has the invariant algebraic surfaces $F_i = 0, i = 1, ..., 6$, given by (see Figures 1 and 2)*

(a) If $b = 2a$ then $F_1 = x^2 - 2az$;
(b) If $a = -b = c$ then $F_2 = y^2 + z^2$;
(c) If $a = b = -c$ then $F_3 = 2x^2 + y^2 + z^2$;
(d) If $3a+c = 0, b = 0$ then $F_4 = x^4 + \frac{4}{3}cx^2z - \frac{4}{9}c^2y^2 - \frac{8}{9}c^2xy - \frac{16}{9}c^2x^2$;
(e) If $a + c = 0, b = 4a$ then $F_5 = x^4 + 4cx^2z - 4c^2y^2 + 8c^2xy + 8c^2x^2 + 48c^3z$;
(f) If $b = c = 0$ then $F_6 = y^2 + z^2 + 2az$.

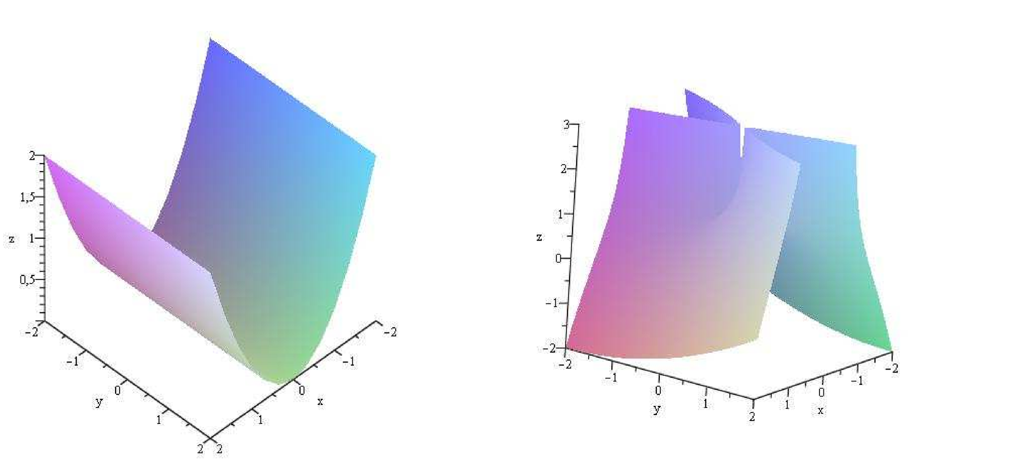

Figure 1. Geometrical representation of the invariant algebraic surfaces of the Chen system: (left) $F_1 = 0$, with $a = 1$; (right) $F_4 = 0$, with $c = 1$.

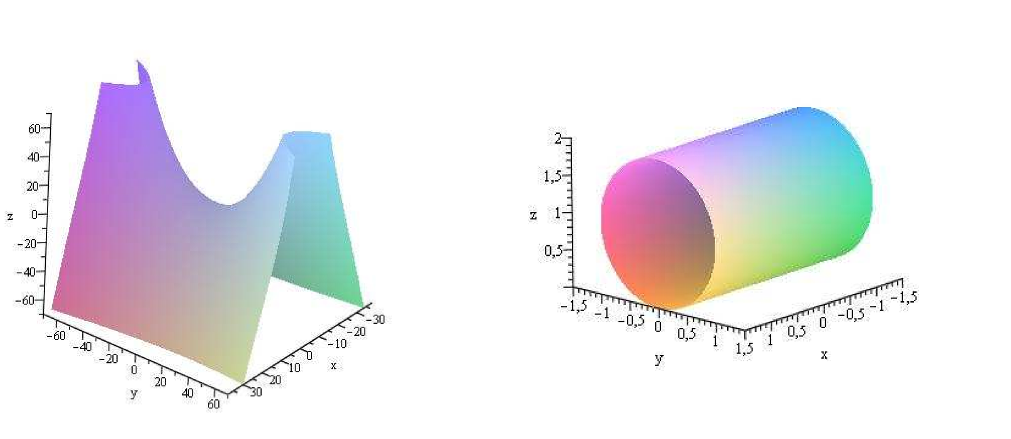

Figure 2. Geometrical representation of the invariant algebraic surfaces of the Chen system: (left) $F_5 = 0$, with $c = 3$; (right) $F_6 = 0$, with $a = -1$.

2.1. Dynamics at Infinity and on the invariant surfaces

By using the Poincaré compactification for a polynomial vector field on $\mathbb{R}^3$ the dynamics of system (1) at infinity was studied and we have obtained the following result.

Theorem 2. *For all values of the parameters a, b, c the phase portrait of system (1) on the Poincaré sphere (at infinity) is as shown in Figure 3: it has two centers at the endpoints of the x–axis, the period annulus of these centers end at the circle defined by the infinity of the plane $\{x = 0\}$, which is filled of singular points.*

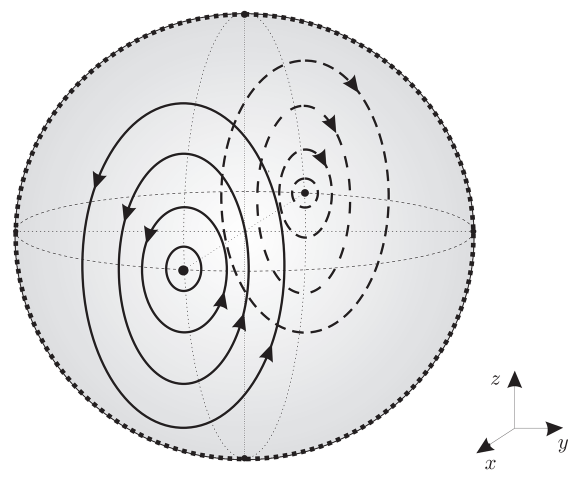

Figure 3. Phase portrait of system (1) at infinity (i.e. on the Poincaré sphere $\mathbb{S}^2$).

The global study of how the invariant algebraic surfaces of the Chen system and the solutions on them reach the infinity, which completes the analysis presented in [4], was done. The following results were obtained.

Theorem 3. *For all values of the parameters a, b, c the z–axis is an invariant set of system (1). The flow in this invariant straight line is as follows: if $b > 0$ then the origin $(0, 0, 0)$ is a global attractor along the z–axis; if $b < 0$ then the origin is a global repeller; and if $b = 0$ all points on the z–axis are singular. Furthermore the global phase portraits of system (1) with $a \neq 0$ and having the invariant algebraic surfaces given in Proposition 1 are described below.*

(a) If $b = 2a$, then system (1) has the invariant algebraic surface $F_1 = x^2 - 2az = 0$. The boundary at infinity of the surface $F_1 = 0$ is the half great circle $\{x = 0, z \leq 0\}$. The finite singular points on $F_1 = 0$ provide 20 different global phase portraits in the Poincaré ball.

(b) If $a = c = -b$, then system (1) has the invariant algebraic surface $F_2 = y^2 + z^2 = 0$, which reduces to the x–axis. The boundary of this surface at infinity is given by the endpoints of the x–axis. The origin is the unique finite singular point, providing 2 trivial different global phase portraits on the Poincaré ball.

(c) If $a = b = -c$, then system (1) has the invariant algebraic surface $F_3 = 2x^2 + y^2 + z^2 = 0$, which reduces to the point $(0, 0, 0)$. The origin is the only finite singular point, providing 2 different global phase portraits, a global attractor and a global repeller, respectively.

(d) If $3a + c = 0, b = 0$, then system (1) has the invariant algebraic surface $F_4 = x^4 + 4/3cx^2z - 4/9c^2y^2 - 8/9c^2xy - 16/9c^2x^2 = 0$, see Figure 1 right. The boundary at infinity of the surface $F_4 = 0$ is the end of the plane $x = 0$. The z–axis is contained in this surface and it is formed by (non-hyperbolici) singular points of the Chen system. The flow restricted to the surface $F_4 = 0$ is integrable. The phase portrait restricted to the invariant algebraic surface $F_4 = 0$ is described in Figure 4.

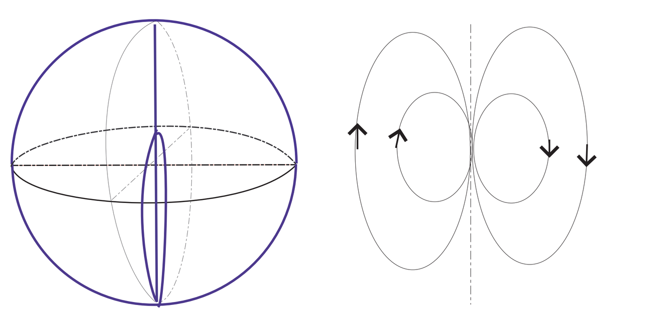

Figure 4. Phase portrait of system (1) on the surface $F_4 = 0$.

(e) If $c = -a$ and $b = 4a$, then system (1) has the invariant algebraic surface $F_5 = x^4 + 4cx^2z - 4c^2y^2 + 8c^2xy + 8c^2x^2 + 48c^3z = 0$, see Figure 2 left. The boundary at infinity of the surface $F_5 = 0$ is the end of the plane $x = 0$. The origin is the only finite singular point, providing 2 different global phase portraits on the Poincaré ball, see Figure 5.

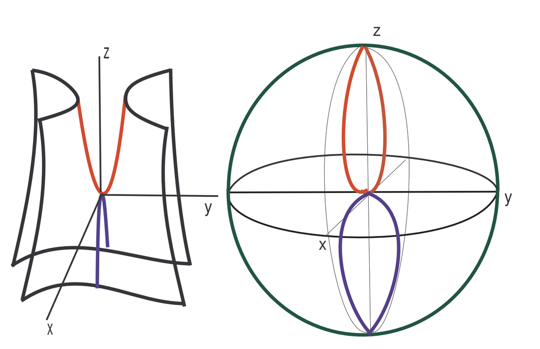

Figure 5. Phase portrait of system (1) on the surface $F_5 = 0$.

(f) Assume that $b = c = 0$, then system (1) has the polynomial first integral $F_6 = y^2 + z^2 + 2az$ and, consequently, $F_6 = constant$ is a family of invariant algebraic surfaces, consisting of enclosed cylinders, see Figure 2 right. The boundary at infinity of the surfaces $F_6 = constant$ is given by the endpoints of the x–axis. The finite singular points are $(0, 0, z)$, with $z \in \mathbb{R}$, that is the z–axis, is a line of (non-hyperbolic) singularities.

2.2. Infinitely many heteroclinic connections

Theorem 4. *Consider system (1) with $a > 0$ and $b = c = 0$. In this case the phase space is foliated by the invariant cylinders*

$$C_\alpha = \{(x, y, z) \mid y^2 + (z + a)^2 = \alpha + a^2, \ x \in \mathbb{R}, \ a > -a^2\} \quad (3)$$

surrounding the line $\{x \in \mathbb{R}, \ y = 0, \ z = -a\}$, each one having two singular points, given by the roots of $z^2 + 2az - \alpha = 0$. One of these singular points is a stable focus (or node) and the other one is a saddle. The 1-dimensional unstable manifolds of the saddle point tend asymptotically to the stable focus (or node), forming two heteroclinic orbits (see Figure 6). Furthermore, the 1-dimensional stable manifolds of the saddle and the orbits contained in the 2-dimensional stable manifold of the focus (or node), except the unstable manifolds of the saddle, are contained in the invariant cylinder C_α and go to infinity in backward time, tending asymptotically to the singular points at infinity, located at the endpoints of the x–axis (see Figure 7). Consequently these singular points at infinity are unstable.

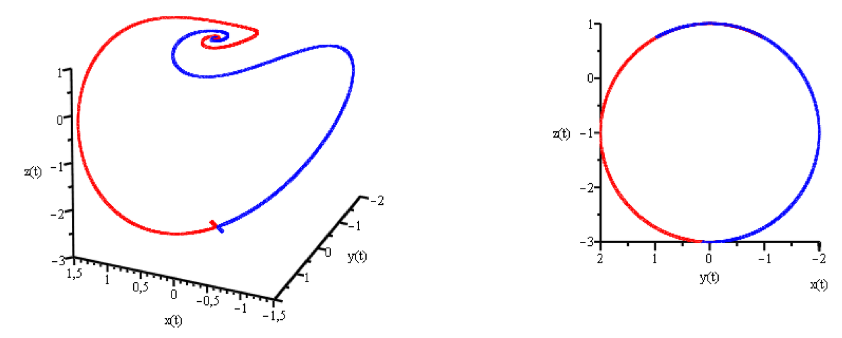

Figure 6. (left) Heteroclinic orbits of the Chen system with $b = c = 0$; (right) its projection onto the yz–plane.

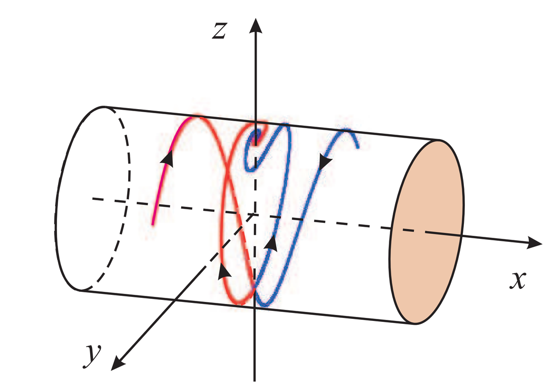

Figure 7. Invariant cylinder containing a saddle and a focus with their invariant manifolds.

From Theorem 4 it follows that system (1) with $a > 0$ and $b = c = 0$ has a family with infinitely many pairs of heteroclinic orbits, each one contained in one of the cylinders C_α given in (3) (see Figures 7–9).

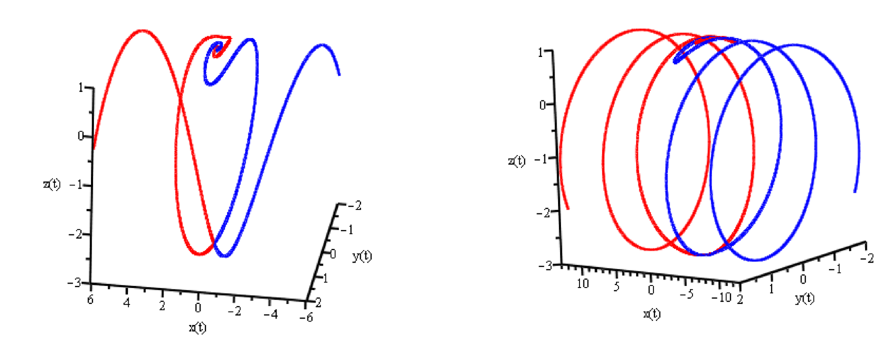

Figure 8. Continuation of Figure 6.

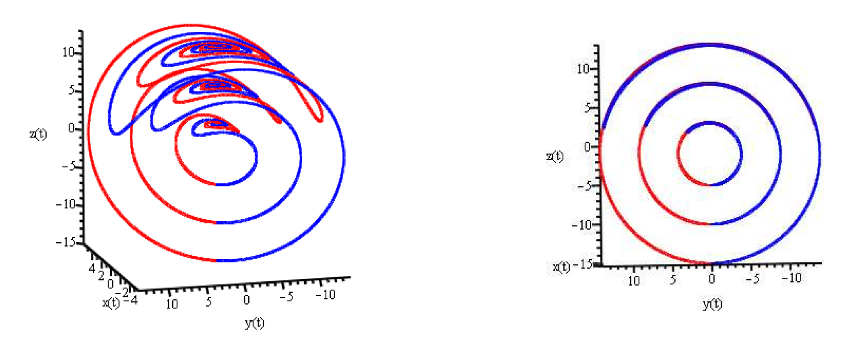

Figure 9. (left) Three pairs of heteroclinic orbits and (right) its projection onto the yz–plane.

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