Limit cycles for a class of non-linear planar piecewise-continuous vector fields

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Introduction

The theory of the non-smooth dynamical systems has been developing very fast in the last years, mainly due to its strong relation with branches of applied science, such as mechanical and aerospace engineering. Indeed, discontinuous systems are in the boundary between mathematics, physics and engineering.

In many cases, the non-smooth systems are described by systems of ODEs in such a way that each system is defined in a region of the phase portrait. The boundary of such regions is called the discontinuity manifold [2].

There are a lot of research being made about the local behavior of non-smooth systems, near the discontinuity manifold [3].

We recall that a limit cycle of a differential system is an isolated periodic orbit in the set of all periodic orbits of the system. In this work we study the existence of limit cycles for the class of discontinuous control systems represented by

$$\begin{cases} \dot{x} = -y + \varepsilon(A_1(x,y) + \phi_0(k_1x + k_2y)b_1), \\ \dot{y} = x + \varepsilon(A_2(x,y) + \phi_0(k_1x + k_2y)b_2), \end{cases} \tag{1}$$

where $(k_1, k_2), (b_1, b_2) \in \mathbf{R}^2$ are nonzero, A_1, A_2 are C^r functions and ϕ_0 is given by

$$\phi_0(z) = \begin{cases} -1 \ , \ z \in (-\infty, 0), \\ \ \ 1 \ \ , \ z \in (0, +\infty). \end{cases}$$

Using the regularization process introduzed by Sotomayor and Teixeira, we approximate system (1) by considering

$$\phi_\omega(z) = \begin{cases} -1 \ , \ z \in (-\infty, -\omega), \\ z/\omega \ , \ z \in [-\omega, +\omega], \\ \ \ 1 \ \ , \ z \in (+\omega, +\infty) \end{cases}$$

and

$$\begin{cases} \dot{x} = -y + \varepsilon(A_1(x,y) + \phi_\omega(k_1x + k_2y)b_1), \\ \dot{y} = x + \varepsilon(A_2(x,y) + \phi_\omega(k_1x + k_2y)b_2), \end{cases} \tag{2}$$

Our main result is the following:

Theorem A: Consider system (2) with A_1, A_2 polynomial functions of degree m. Then for every small $\varepsilon > 0$, at most $\dfrac{m-1}{2}$ limit cycles bifurcates from the periodic orbits of system (2) with $\varepsilon = 0$. Moreover, there are systems having exactly $\dfrac{m-1}{2}$ limit cycles.

The proof of Theorem A relies on Averaging Theory. We mention that a similar result holds for higher dimensional systems, considering A_1, A_2 linear [4].

Averaging Theory

The first-order averaging theory developed in [1] is summarized as follows.

Theorem 1. *Consider the differential system*

$$\dot{x}(t) = \varepsilon F_1(t,x) + \varepsilon^2 R(t,x,\varepsilon), \tag{3}$$

where $F_1 : \mathbb{R} \times D \to \mathbb{R}^n$, $R : \mathbb{R} \times D \times (-\varepsilon_f, \varepsilon_f) \to \mathbb{R}^n$ are continuous functions, T–periodic in the first variable, and D is an open subset of $\mathbb{R}^n$. We define $F_{10} : D \to \mathbb{R}^n$ as

$$F_{10}(z) = \frac{1}{T}\int_0^T F_1(s,z)ds,$$

and we assume that the following hypotheses (i) and (ii) hold.

(i) F_1 and R are locally Lipschitz with respect to x;

(ii) For $a \in D$ with $F_{10}(a) = 0$, there exists a neighborhood V of a such that $F_{10}(z) \neq 0$ for all $z \in \bar{V} \backslash \{a\}$ and $d_B(F_{10}, V, a) \neq 0$.

Then for $|\varepsilon| > 0$ sufficiently small there exists a T–periodic solution $\psi(\cdot, \varepsilon)$ of system (3) *such that $\psi(0,\varepsilon) \to a$ as $\varepsilon \to 0$.*

The expression $d_B(F_{10}, V, a_\varepsilon) \neq 0$ means that the Brouwer degree of the function $F_{10} : V \to \mathbb{R}^n$ at the fixed point a is not zero.

Proof of Theorem A

We apply Theorem 1 to system (2). We fix m odd (the case m even can be discarded)

$$A_1(x,y) = \sum_{k=1}^m \sum_{a_{i,j}x^iy^j}, \ A_2(x,y) = \sum_{k=1}^m \sum_{b_{i,j}x^iy^j}.$$

The change of coordinates $x = r\cos(\theta)$, $y = r\sin(\theta)$ turns system (2) into

$$\begin{cases} \dot{r} = \varepsilon F_\omega(r,\theta) \\ \dot{\theta} = 1 + \varepsilon G_\omega(r,\theta), \end{cases} \tag{4}$$

where F_ω, G_ω depends on A_1, A_2. We take θ as the new time, so the new system write as

$$\frac{dr}{d\theta} = \varepsilon F_\omega(r,\theta) + \varepsilon^2 R_\omega(r,\theta,\varepsilon),$$

where R_ω depends on F_ω, G_ω and. The system above is the form of (**??**).

To compute the averaged system, we have to take into account if $r < \omega$ of $r \geq \omega$. As $\omega \to 0$, the the case $r < \omega$ is not interesting for us.

Computing the averaged equation for $r \geq \omega$, we obtain

$$\begin{aligned} \dot{r} = \varepsilon H(r,\omega) &= \varepsilon\left(\frac{2}{r}b_1\sqrt{r^2-\omega^2} - \frac{2b_1r}{\omega}\arccos\left(\frac{\omega}{r}\right)\right) \\ &+ \varepsilon\left(\sum_{k=1}^{\frac{m-1}{2}} \lambda_{2j+1}r^{2j+1}\right), \end{aligned} \tag{5}$$

where $\lambda_1 = \dfrac{b_1}{\omega} + a_{1,0} + b_{0,1}$ and for $k = 2l+1$, $l \geq 1$,

$$\lambda_k = \rho_k\left(\sum_{i=1}^{l} a_{2i+1,k-(2i+1)} + b_{k-(2i+1),2i+1}\right) + \sigma_k(a_{k,0} + b_{0,k}),$$

and ρ_k, σ_k are positive constants depending on k. By Theorem 1, each simple zero r_ω of system (5) such that $\lim_{\omega\to 0} r_\omega \neq 0$ is related to a limit cycle of system (2) (and system (1)) for each $\varepsilon > 0$ small.

After some calculations, we can show that is possible to find $\lambda_1, \ldots, \lambda_m$ so that $H(r,\omega)$ has l (simple) zeroes, for l from 1 to $\frac{m-1}{2}$.

Now we provide examples with 1 and $\frac{m-1}{2}$ limit cycles.

Example 1: Take $\lambda_j = 1$, for all j, and $b_1 = 1$. In this case, the polynomial part and the negative of the terms involving square roots and $\arccos$ in $H(r,\omega)$ are strictly convex for $r > 0$. This example works for every m.

Example 2: To obtain the $\frac{m-1}{2}$ limit cycles, we have to interpolate the polynomial part of $H(r,\omega)$. We fix $m = 7$. Take $b_1 = \lambda_7 = 1$ and

$$\begin{aligned} \lambda_1 &= -576\,\omega^6 - \frac{1}{2520}\,\frac{-4032\,\pi + 3024\,\sqrt{3} - 2160\ \arccos(1/4) + 135\,\sqrt{15} + 9216\ \arccos(1/3) - 2048\,\sqrt{2} + 2520}{\omega} \\ \lambda_3 &= 244\,\omega^4 + \frac{1}{2016}\,\frac{-560\,\pi + 420\,\sqrt{3} - 624\ \arccos(1/4) + 39\,\sqrt{15} + 2304\ \arccos(1/3) - 512\,\sqrt{2}}{\omega^3} \\ \lambda_5 &= -29\,\omega^2 - \frac{1}{10080}\,\frac{-112\,\pi + 84\,\sqrt{3} - 240\ \arccos(1/4) + 15\,\sqrt{15} + 576\ \arccos(1/3) - 128\,\sqrt{2}}{\omega^5} \end{aligned}$$

With this values for λ_j, we obtain exactly 3 limit cycles.

References

[1] A. Buică and J. Llibre, *Averaging methods for finding periodic orbits via Brouwer degree*, Bull. Sci. Math. **128** (2004), 7–22.

[2] J. Llibre, P. R. da Silva, M. A. Teixeira, *Study of singularities in nonsmooth dynamical systems via singular perturbation*. SIAM J. Appl. Dyn. Syst. **8** (2009), 508–526.

[3] M. Guardia, T. Seara, M. A. Teixeira, *Generic bifurcations of low codimension of planar Filippov systems*. J. Differential Equations **250** (2011), 1967–2023.

[4] M. Lima, J. Llibre, *Limit cycles and invariant cylinders for a class of continuous and discontinuous vector field in dimention $2n$*. Applied Mathematics and Computation **217** (2011), 9985–9996.

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