The Specification Property in the Dynamics of Linear Operators

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Introduction

Linear dynamics is mainly concerned with the behaviour of iterates of linear transformations. In an infinite-dimensional setting linear operators may have dense orbits. If T is a continuous linear operator acting on some topological vector space X, T is said to be hypercyclic if there exists some vector whose T-orbit is dense in X. A first characterization of hypercyclicity is a direct application of the Baire category theorem. On a separable Banach space, hypercyclicity is equivalent to topological transitivity. The mixing property is a strong form of topological transitivity. We also consider two qualitative strengthenings of hypercyclicity, namely chaoticity and frequent hypercyclicity.

We introduce the notion of the Specification Property (SP) for operators on Banach spaces, inspired by the usual one of Bowen for continuous maps on compact spaces. This is a very strong dynamical property related to the chaotic behaviour. Several general properties of operators with the SP are established. For instance, every operator with the SP is mixing, Devaney chaotic, and frequently hypercyclic. In the context of weighted backward shifts, the SP is equivalent to Devaney chaos. In contrast, there are Devaney chaotic operators (respectively, mixing and frequently hypercyclic operators) which do not have the SP.

Devaney chaos and mixing properties have been widely studied for linear operators on Banach and more general spaces. The recent books [7, 4] contain the basic theory, examples and many results on chaotic linear dynamics. Some works on the specification properties are [9, 8].

1. The specification property. Basic properties

A continuous map on a metric space is said to be chaotic in the sense of Devaney if it is topologically transitive and the set of periodic points is dense. A notion of chaos stronger than Devaney's definition is the so called specification property. Roughly speaking a continuous map on a compact metric space satisfies the specification property if one can approximate simultaneously distinct pieces of orbits by one periodic orbit. In other words, one can specified the orbit of some x concatenating arbitrary sequences of pieces of orbits. The specification property was introduced by Bowen [5]; since then, several kinds and degrees of this property have been stated. We will follow the definitions and terminology used in [3].

Definition 1. *A continuous map $f : X \to X$ on a compact metric space (X, d) has the* **strong specification property** *if for any $\delta > 0$ there is a positive integer N_δ such that for any integer $s \geq 2$, any set $\{y_1, \dots, y_s\} \subset X$ and any integers $0 = j_1 \leq k_1 < j_2 \leq k_2 < \cdots < j_s \leq k_s$ satisfying $j_{r+1} - k_r \geq N_\delta$ for $r = 1, \dots, s-1$, there is a point $x \in X$ such that, for each positive integer $r \leq s$ and all integers i with $j_r \leq i \leq k_r$, the following conditions hold:*

$$d(f^i(x), f^i(y_r)) < \delta,$$

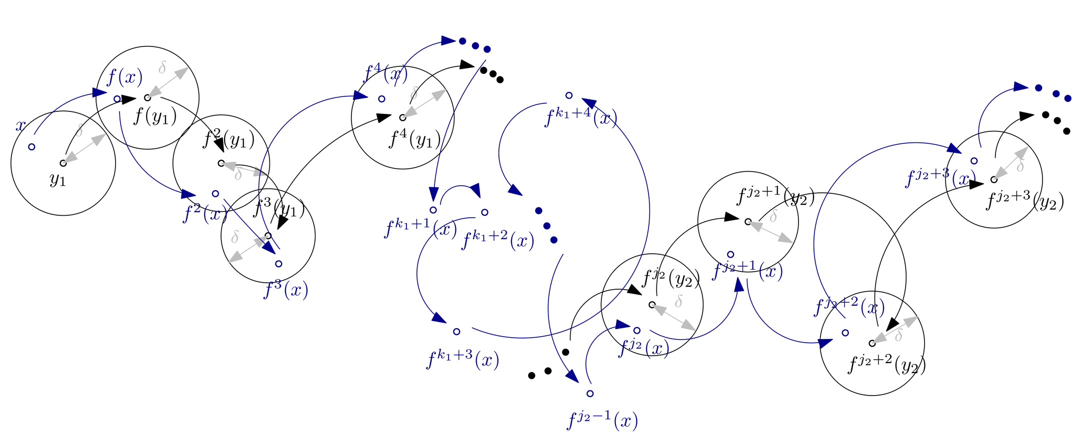
$$f^n(x) = x, \quad \text{where} \quad n = N_\delta + k_s.$$

We plan to study this strong specification property for bounded linear operators defined on separable Banach spaces. In this situation, the first crucial problem is that this spaces are never compact. Next definition states this property in this setting.

Definition 2. *A bounded linear operator $T : X \to X$ on a separable Banach space X has the* **specification property** *(SP) if there exists an increasing sequence $(K_m)_m$ of T-invariant compact sets with $0 \in K_1$ and $\overline{\cup_{m\in\mathbb{N}} K_m} = X$ such that for each $m \in \mathbb{N}$ the map $T|_{K_m}$ has the SP, that is, for any $\delta > 0$ there is a positive integer $N_{\delta,m}$ such that for every $s \geq 2$, any set $\{y_1, \dots, y_s\} \subset K_m$ and any integers $0 = j_1 \leq k_1 < j_2 \leq k_2 < \cdots < j_s \leq k_s$ with $j_{r+1} - k_r \geq N_{\delta,m}$ for $1 \leq r \leq s-1$, there is a point $x \in K_m$ such that, for each positive integer $r \leq s$ and integers i with $j_r \leq i \leq k_r$, the following conditions hold:*

$$\left\| T^i(x) - T^i(y_r) \right\| < \delta,$$

$$T^n(x) = x, \quad \text{where } n = N_{\delta,m} + k_s.$$



Proposition 3. *Let $T : X \to X$ be a bounded operator on a separable Banach space X. If T has the specification property then so does T^n for all $n \in \mathbb{N}$.*

Proposition 4. *Suppose $T_i : X_i \to X_i$ is a bounded operator on a separable Banach space X_i, $i = 1, 2$, and $\phi : X_1 \to X_2$ is a continuous map with dense range such that*

$$\begin{array}{ccc} X_1 & \xrightarrow{T_1} & X_1 \\ \phi\downarrow & & \downarrow\phi \\ X_2 & \xrightarrow[T_2]{} & X_2 \end{array}$$

conmutes. If T_1 has the SP then so does T_2.

Proposition 5. *Let $T_i : X_i \to X_i$ be a bounded operator on a separable Banach space X_i, for $i = 1, 2$. If T_1 and T_2 have the SP then so does $T_1 \oplus T_2$.*

Proposition 6. *Assume that $T : X \to X$ is a bounded operator on a separable Banach space X. If $\lambda_i \in \mathbb{K}$ and $K_i \subset X$ is a T-invariant subset such that $T|_{K_i}$ has the SP, $i = 1, \dots, l$, then $T|_{\sum_{i=1}^{l} \lambda_i K_i}$ has the SP.*

Proposition 7. *Suppose that $T : X \to X$ is a bounded operator on a separable Banach space X. If $K \subset X$ is a T-invariant set such that $T|_K$ has the SP and $\mathrm{co}(K)$ is the convex envelop of K, then $T|_{\mathrm{co}(K)}$ has the SP.*

2. Connections with other dynamical properties

Definition 8. *T is* **topologically transitive** *if, for any $U, V \subset X$ non-empty open sets there exists $n \in \mathbb{N}$ such that $T^n(U) \cap V \neq \emptyset$.*

Within our context, this is equivalent to **hypercyclicity**, that is, the existence of vectors $x \in X$ whose orbit under T is dense in X.

Definition 9. *T is* **topologically mixing** *if for any $U, V \subset X$ non-empty open sets there exists $N \in \mathbb{N}$ such that $T^n(U) \cap V \neq \emptyset$ for all $n \geq N$.*

Proposition 10. *If T has the SP then T is mixing and hence hypercyclic.*

For the proof we use the so called return sets: Given any non-empty open sets U and V the return set is defined as

$$N(U, V) := \{n \in \mathbb{N} \; : \; T^n(U) \cap V \neq \emptyset\}.$$

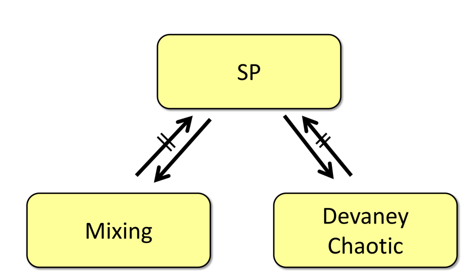
It is clear that T is mixing if and only if $N(U, V)$ is cofinite for any non-empty open sets U and V. We prove that $N(U, W)$ and $N(W, V)$ are cofinite, where W is an open 0-neighbourhood.

Definition 11. *T is* **chaotic** *in the sense of Devaney if*

1. *T is topologically transitive, and*
2. *the set $\mathrm{Per}(T) := \{$periodic points of $T\} = \{x \in x \; ; \; T^n x = x$ for some $n\}$ is dense in X.*

Proposition 12. *If T has the SP then T is chaotic in the sense of Devaney.*

Remark 13. *There are easy examples of mixing but not Devaney chaotic operators and, on the other hand, Badea and Grivaux constructed Devaney chaotic operators which are not mixing. Therefore, neither the mixing property nor Devaney chaos implies the SP.*



As noted before, an operator T is hypercyclic if there is some vector $x \in X$ whose orbit is dense. This means that for any non-empty open set U the set

$$N(x, U) := \{n \in \mathbb{N} \; : \; T^n(x) \in U\}$$

is infinite. One may ask how often the orbit of x visits each open set U. This question lead us to the notion of frequent hypercyclicity.

Definition 14. *T is* **frequently hypercyclic** *if there is $x \in X$ such that, for each non-empty open set $U \subset X$,*

$$\underline{\mathrm{dens}}(N(x, U)) := \liminf_n \frac{\left|\{k \leq n \; ; \; T^k x \in U\}\right|}{n} > 0.$$

Theorem 15. *If T has the SP then T is frequently hypercyclic.*

The proof consists in the construction of a vector (which is not trivial) frequently hypercyclic for T. For more details we refer to [2]

Remark 16. *If $A \subset \mathbb{N}$ is a n infinite set which contains the least common multiple of every pair of elements in A, then there is an operator T_A on ℓ^2 that is frequently hypercyclic, mixing and chaotic, and such that its set of periods coincides with A (see [6]). Therefore, if we select for instance $A = \{2^m\}_m$, the operator T_A cannot satisfy the SP. In other words,*

$$\boxed{(\text{ mixing } + \text{ Dev. chaotic } + \text{ freq. hyp }) \not\Rightarrow \text{ SP}}$$

3. The backward shift operator

In this last section we characterize SP for a class of operators, namely, backward shift operators defined on sequence spaces (see [1]). For a strictly positive sequence of weights $v = (v_n)_n$, consider the Banach de sequence spaces

$$\ell^p(v) = \left\{ (x_n)_n \; ; \; \|x\| := \left(\sum_{n=1}^{\infty} |x_n|^p v_n \right)^{1/p} < \infty \right\}, \; 1 \leq p < \infty,$$

$$c_0(v) = \left\{ (x_n)_n \; ; \; \lim_{n\to\infty} |x_n| v_n = 0, \; \|x\| := \sup_n |x_n| v_n \right\}$$

The **backward shift** B is defined as

$$B(x_1, x_2, x_3, \dots) := (x_2, x_3, x_4, \dots)$$

B is a bounded operator iff $\sup_{n\in\mathbb{N}} \frac{v_n}{v_{n+1}} < \infty$.

Theorem 17. *For the backward shift operator B on $\ell^p(v)$, $1 \leq p < \infty$, (respectively, on $c_0(v)$) the following conditions are equivalent:*

(i) $\sum_{n=1}^{\infty} v_n < \infty$ (respectively, $\lim_{n\to\infty} v_n = 0$).

(ii) B has the SP.

(iii) B is Devaney chaotic.

Theorem 18. *For a sequence $w = (w_n)_n$ of non-zero scalars, the associated* **weighted backward shift** *is defined as*

$$B_w(x_1, x_2, \dots) := (w_2 x_2, w_3 x_3, \dots)$$

If B_w is a bounded operator on $\ell^p(v)$, $1 \leq p < \infty$, then the following conditions are equivalent:

(i) $\sum_{n=1}^{\infty} \frac{v_n}{\prod_{i=1}^{n} |w_i|^p} < \infty$.

(ii) B_w has the SP.

(iii) B_w is Devaney chaotic.

References

[1] S. Bartoll, F. Martínez-Giménez and A. Peris, *The specification property for backward shifts*, Journal of Difference Equations and Applications (2011), pp 1-7.

[2] S. Bartoll, F. Martínez-Giménez and A. Peris, *The specification property in the dynamics of linear operators*. Preprint.

[3] W. Bauer and K. Sigmund, *Topological dynamics of transformations induced on the space of probability measures*. Monatsh. Math. **79** (1975), pp. 81–92.

[4] F. Bayart and É. Matheron, *Dynamics of Linear operators*. Cambridge University Press, 2009.

[5] R. Bowen. *Topological entropy and axiom* A. Global Analysis (Proc. Sympos. Pure Math., Vol. XIV, Berkeley, Calif., 1968), Amer. Math. Soc., Providence, R.I., (1970), pp. 23–41.

[6] J. A. Conejero, F. Martínez-Giménez and A. Peris. *Sets of periods for chaotic linear operators*. Preprint 2012.

[7] K. G. Grosse-Erdmann and A. Peris, *Hypercyclicity and linear chaos*. Springer, 2011.

[8] P. Oprocha. *Specification properties and dense distributional chaos*. Discr. Contin. Dyn. Syst. 17 (2007), pp. 821 - 833.

[9] K. Sigmund, *On dynamical systems with the specification property*. Trans. Amer. Math. Soc. **190** (1974), pp. 285–299.