Limit Cycles for a class of continuous piecewise linear differential systems with three zones

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Introduction

Due to the encouraging increase in their applications, control theory [3], design of electric circuits [1], neurobiology [2] piecewise linear differential systems were studied early from the point of view of qualitative theory of ordinary differential equations. Nowadays, a lot of papers are being devoted to these differential systems.

In this work we study the existence of limit cycles for the class of continuous piecewise linear differential systems

$$\mathbf{x}' = X(\mathbf{x}), \tag{1}$$

where $\mathbf{x} = (x, y)$ and X is a continuous piecewise linear v.f. We will consider the following situation: we have two parallel straight lines L_- and L_+ symmetric with respect to the origin dividing the phase plane in three closed regions: R_-, R_o and R_+ with $(0,0) \in R_o$ and the regions R_- and R_+ have as boundary the straight lines L_- and L_+ respectively. We will denote by X_- the vector field X restrict to R_-, by X_o the vector field X restricted to R_o and by X_+ the vector field X restrict to R_+. We suppose that the restriction of the vector field to each one of these zones are linear systems with constant coefficients that are glued continuously at the common boundary.

In short, system (1) can be written as

$$\mathbf{x}' = \begin{cases} A_-\mathbf{x} + B_- & \mathbf{x} \in R_-, \\ A_o\mathbf{x} + B_o & \mathbf{x} \in R_o, \\ A_+\mathbf{x} + B_+ & \mathbf{x} \in R_+, \end{cases} \tag{2}$$

where $A_i \in \mathcal{M}_2(\mathcal{R})$, $i \in \{-, o, +\}$, $B_i \in \mathcal{R}^2$, $i \in \{-, o, +\}$ and $\mathbf{x}' = \dfrac{d\mathbf{x}}{dt}$ with t the time. In what follows we denote by $t_i = trace(A_i)$ and by $d_i = det(a_i)$ for $i \in \{-, o, +\}$.

We say that the vector field X has a real equilibrium x^* in R_i with $i \in \{-, o, +\}$ if x^* is an equilibrium of X_i and $x^* \in R_i$. In opposite we will say that X has a virtual equilibrium x^* in R_i if $x^* \in R_i^c$.

We suppose the following assumptions:

(H1) X_o has a real equilibrium in the interior of the region R_o of focus type.

(H2) The others equilibria (real or virtual) of X_- and X_+ are a center and a focus with different stability with respect to the focus of X_o.

Our main result is the following.

Theorem A: Assume that system (2) satisfies assumptions $(\mathrm{H1})$ and $(\mathrm{H2})$. Then system (2) has a unique limit cycle, which is hyperbolic .

1. Normal Form

In this section we will write system (2) in a convenient normal form where the number of parameters are reduced.

Lemma 1: Suppose that system (2) is such that $det(A_o) > 0$. Then there exists a linear change of coordinates that writes system (2) into the form $\dot{\mathbf{x}} = X(\mathbf{x})$, with $L_- = L_{-1} = \{x = -1\}$, $L_+ = L_1 = \{x = 1\}$, $R_- = \{(x, y) \in \mathcal{R}^2;\ x \le -1\}$, $R_o = \{(x, y) \in \mathcal{R}^2;\ -1 \le x \le 1\}$, $R_+ = \{(x, y) \in \mathcal{R}^2;\ x \ge 1\}$ and

$$X(\mathbf{x}) = \begin{cases} A_-\mathbf{x} + B_- & \mathbf{x} \in R_-, \\ A_o\mathbf{x} + B_o & \mathbf{x} \in R_o, \\ A_+\mathbf{x} + B_+ & \mathbf{x} \in R_+, \end{cases} \tag{3}$$

where $A_- = \begin{pmatrix} a_{11} & -1 \\ 1 - b_2 + d_2 & a_1 \end{pmatrix}$, $B_- = \begin{pmatrix} a_{11} \\ d_2 \end{pmatrix}$, $A_o = \begin{pmatrix} 0 & -1 \\ 1 & a_1 \end{pmatrix}$, $B_o = \begin{pmatrix} 0 \\ b_2 \end{pmatrix}$, $A_+ = \begin{pmatrix} c_{11} & -1 \\ 1 + b_2 - f_2 & a_1 \end{pmatrix}$ and $B_+ = \begin{pmatrix} -c_{11} \\ f_2 \end{pmatrix}$. The dot denotes derivative with respect to a new time s.

For our purpose we will define a first return map that involves all the vector fields X_-, X_o and X_+ and the transversal sections L_- and L_+. For this we need the next Lemma.

Lemma 2: In the coordinates given by Lemma 1 there is a unique contact point of system (3) with L_- and a unique contact point of (3) with L_+. These points are respectively $p_- = (-1, 0)$ and $p_+ = (1, 0)$. Moreover under the assumptions (H1) and (H2) the equilibria of X_- and X_+ are virtual.

2. Poincaré Return Map

Under the assumptions (H1) and (H2) we have that system (3) has a unique real equilibrium in R_o and, by Lemma 2, the two other equilibria are virtual.

In order to study the existence of limit cycles for system (3) under the assumptions H1 and H2 we will define a Poincaré return map defined on L_-.

Let p_- be the contact point of $\dot{\mathbf{x}} = A_-\mathbf{x} + B_-$ with L_-. Note that p_- divides L_- into two segments L_-^O and L_-^I where in L_-^O the vector field points toward the region R_- while in L_-^I the vector field points toward the region R_o. In fact we have $L_-^O = \{(-1, y);\ y \ge 0\}$ and $L_-^I = \{(-1, y);\ y \le 0\}$.

We can define a Poincaré map $\Pi_- : L_-^O \to L_-^I$ by $\Pi_-(p) = q$ as been the first return map in forward time of the flow of $\dot{\mathbf{x}} = A_-\mathbf{x} + B_-$ to L_-. Observe that $\Pi_-(p_-) = p_-$.

Note that we can see the mapping Π_- in a different way. Given, $p \in L_-^O$ and $q \in L_-^I$ there exist unique $a \ge 0$ and $b \ge 0$ such that $p = p_- - a\dot{p}_-$ where $\dot{p}_- = X_-(p_-) = (0, b_2 - 1)$, and $q = p_- + b\dot{p}_-$. So the mapping Π_- induces a mapping π_- given by $\pi_-(a) = b$.

Proposition 3: Consider the vector field X_i in R_i with $i \in \{-, +\}$, with a virtual center or focus equilibrium and such that $t_i \ge 0$ (resp. $t_i < 0$). Let π_i be the map associated to the Poincaré map $\Pi_i : L_i \to L_i$ defined by the flow of the linear system $\dot{\mathbf{x}} = A_i\mathbf{x} + B_i$.

(a) If $t_i > 0$ (resp. $t_i < 0$) then the maps π_i satisfy that $\pi_i : [0, \infty) \to [0, \infty)$, $\pi_i(0) = 0$, $\lim_{a \to \infty} \pi_i(a) = +\infty$ and $\pi_i(a) > a$ (resp. $\pi_i(a) < a$) in $(0, \infty)$.

(a.1) If $a \in (0, \infty)$ then $(\pi_i)'(a) = \dfrac{a}{\pi_i(a)} e^{2\gamma_i \tau_i}$. Moreover $(\pi_i)'(a) > 1$ (resp. $0 < (\pi_i)'(a) < 1$) and $\lim_{a \to 0} (\pi_i)'(a) = 1$.

(a.2) If $a \in (0, \infty)$ then $(\pi_i)''(a) > 0$ (resp. $(\pi_i)''(a) < 0$).

(a.3) The straight line $b = e^{\gamma_i \pi} a - t_i(1 + e^{\gamma_i \pi})/d_i$ (resp. $b = e^{\gamma_i \pi} a - t_i(1 + e^{-\gamma_i \pi})/d_i$) in the plane (a, b) is an asymptote of the graph of π_i when a tends to $+\infty$ where $\gamma_i = t_i / \sqrt{4d_i - t_i^2}$. So $\lim_{a \to \infty} (\pi_1)'(a) = e^{\gamma_i \pi}$.

(b) If $t_i = 0$ then π_i is the identity in $[0, \infty)$.

Let p_- and p_+ be the contact point of $\dot{\mathbf{x}} = A_o\mathbf{x} + B_o$ with L_- and L_+ respectively. We can define a Poincaré map $\Pi_o : D_o^* \subset L_- \to L_+$ by $\Pi_o(q) = r$ being the map from points in D_o to points in L_+ defined by the flow of $\dot{\mathbf{x}} = A_o\mathbf{x} + B_o$ in forward time, where D_o^* is a subset of L_- where the mapping Π_o is well defined.

As before we can see the mapping Π_o in a different way. Given, $q \in D_o^*$ and $r \in L_+$ there exist unique $b \ge 0$ and $c \ge 0$ such that $q = p_- + b\dot{p}_-$ and $r = p_+ - c\dot{p}_+$, where $\dot{p}_+ = X_+(p_+) = (0, b_2 + 1)$. So the mapping Π_o induces a mapping π_o given by $\pi_o(b) = c$. As before we will consider the map π_o instead of Π_o.

In the same way we can define a first return map $\bar{\Pi}_o : \bar{D}_o^* \subset L_+ \to L_-$ and the respective $\bar{\pi}_o$.

The next propositions state the qualitative behavior of these maps.

Proposition 4: Consider the vector field X_o in R_o with a real focus equilibrium and such that $t_o > 0$ (resp. $t_o < 0$). Let π_o be the map associated to the Poincaré map $\Pi_o : D_o^* \subset L_- \to L_+$ defined by the flow of the linear system $\dot{\mathbf{x}} = A_o\mathbf{x} + B_o$ from the straight line L_- to the straight line L_+.

(a) If $0 < b_2 < 1$ (resp. $-1 < b_2 < 0$) then the map π_o satisfies that $\pi_o : [b^*, \infty) \to [c^*, \infty)$, $b^*, c^* \ge 0$ with $\pi_o(b^*) = c^*$ and $\lim_{b \to \infty} \pi_o(b) = +\infty$. Moreover $b^* = 0$ if and only if $e^{\gamma_o \pi} \ge \dfrac{1 + b_2}{1 - b_2}$ and $c^* = 0$ if and only if $e^{\gamma_o \pi} \le \dfrac{1 + b_2}{1 - b_2}$.

(a.1) If $b \in (b^*, \infty)$ then $\pi_o'(b) = \left(\dfrac{1 - b_2}{1 + b_2}\right)^2 e^{2\gamma_o \tau_o} \dfrac{b}{\pi_o(b)}$, with $\tau_o \to 0$ when $b \to \infty$ and $\lim_{b \to \infty} \pi_o'(b) = \left(\dfrac{1 - b_2}{1 + b_2}\right)^2$.

(b) If $-1 < b_2 \le 0$ (resp. $0 \le b_2 < 1$) then $b^* = 0$ (resp. $c^* = 0$) and π_o satisfies $\pi_o : [0, \infty) \to [c^*, \infty)$, $c^* > 0$ with $\pi_o(0) = c^*$ (resp. $\pi_o : [b^*, \infty) \to [0, \infty)$, $b^* > 0$ with $\pi_o(b^*) = 0$) and $\lim_{b \to \infty} \pi_o(b) = +\infty$.

(b.1) $\pi_o'(b)$ satisfy the same conditions described in (a.1).

Proposition 5: Consider the vector field X_o in R_o with a real focus equilibrium and such that $t_o > 0$ (resp. $t_o < 0$). Let $\bar{\pi}_o$ be the map associated to the Poincaré map $\bar{\Pi}_o : \bar{D}_o^* \subset L_+ \to L_-$ defined by the flow of X_o from the straight line L_+ to the straight line L_-.

(a) If $-1 < b_2 < 0$ (resp. $0 \le b_2 < 1$) then $\bar{\pi}_o$ satisfies that $\bar{\pi}_o : [d^*, \infty) \to [a^*, \infty)$, $d^*, a^* \ge 0$ with $\bar{\pi}_o(d^*) = a^*$ and $\lim_{d \to \infty} \bar{\pi}_o(d) = +\infty$. Moreover $d^* = 0$ if and only if $e^{\gamma_o \pi} \ge \dfrac{1 - b_2}{1 + b_2}$ and $a^* = 0$ if and only if $e^{\gamma_o \pi} \le \dfrac{1 - b_2}{1 + b_2}$.

(a.1) If $d \in (d^*, \infty)$ then $\bar{\pi}_o'(d) = \left(\dfrac{1 + b_2}{1 - b_2}\right)^2 e^{2\gamma_o \bar{\tau}_o} \dfrac{d}{\bar{\pi}_o(d)}$ with $\bar{\tau}_o \to 0$ when $d \to \infty$ and $\lim_{d \to 0} \bar{\pi}_o'(d) = \infty$ and $\lim_{d \to \infty} \bar{\pi}_o'(d) = \left(\dfrac{1 + b_2}{1 - b_2}\right)^2$.

(b) If $0 \le b_2 < 1$ (resp. $-1 < b_2 < 0$) then $d \in (0, \infty)$ and $\bar{\pi}_o$ satisfies $\bar{\pi}_o : [0, \infty) \to [a^*, \infty)$, $a^* > 0$ with $\bar{\pi}_o(0) = a^*$ and $\lim_{d \to \infty} \bar{\pi}_o(d) = +\infty$.

(b.1) $\bar{\pi}_o'(d)$ satisfy the same conditions described in (a.1).

3. Limit Cycles having a Focus in R_o

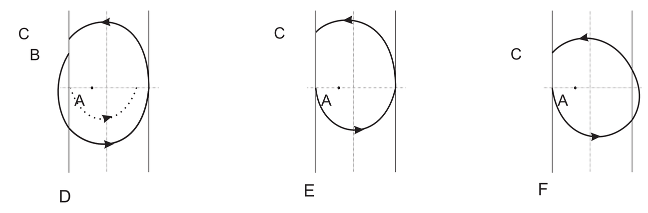
In what follows without loss of generality we suppose that the center of hypothesis (H2) is in R_-. The next two Propositions state **Theorem A**.

Proposition 6: Assume that system (3) satisfies assumptions $(\mathrm{H1})$ and $(\mathrm{H2})$. Suppose that the real equilibrium point in R_o is between L_- and L_o where L_o is a line parallel to L_+ through the origin. Then there exists a unique limit cycle of (3), which is hyperbolic. Moreover this limit cycle visits the three regions R_-, R_o and R_+. It is a repeller if $t_o < 0$, and an attractor if $t_o > 0$.

Proof: Suppose that we have a center in X_-, and a focus in X_o and in X_+. By the hypotheses we have $0 \le b_2 < 1$.

Using the previous notation we have $\gamma_i = \dfrac{\alpha_i}{\beta_i}$, $i \in \{-, o, +\}$. So $\gamma_- = 0$ and $\gamma_o, \gamma_+ \ne 0$. The domain of the first return map Π defined by $\Pi = \bar{\pi}_o \circ \pi_+ \circ \pi_o \circ \pi_-$ is an interval of $\mathcal{R}^+$ that depends on the domain of the mapping π_o and $\bar{\pi}_o$ stated in Propositions 4 and 5.

Suppose that $\gamma_o > 0$ and $\gamma_+ < 0$. In this case $\bar{D}_o(\bar{\pi}_o) = [0, \infty)$ and $D_o(\pi_o) = [b^*, \infty)$ where $b^* \ge 0$ and $\pi_o(b^*) = 0$ (see figure below). This implies that $D(\Pi) = [a^*, \infty)$ where $a^* = b^*$. Moreover we have $\Pi : [a^*, \infty) \to [a^{**}, \infty)$ with $a^{**} > a^*$ and $\Pi(a^*) = a^{**}$



where $A = -b_2$, $B = p_- - a^* X_-(p_-)$, $C = p_- - a^{**} X_-(p_-)$, $D := \pi(a^*) = a^{**}, E, F := \pi(0) = a^{**}$.

Define the displacement function $h(a) = \Pi(a) - a$.

Note that finding a fixed point of Π is equivalent to find zeroes of the function h.

In a^* we have $h(a^*) = \Pi(a^*) - a^* > 0$. Supposing that h admits a zero and that a_s is the smallest zero we must have $h'(a_s) \le 0$, or equivalently $\Pi'(a_s) \le 1$. But from the definition of Π we can write

$$\Pi'(a) = \bar{\pi}_o'(d)\pi_+'(c)\pi_o'(b)\pi_-'(a)$$

where $b = \pi_-(a) = a$, $c = \pi_o(b)$ and $d = \pi_+(c)$.

From Propositions 3-6 it follows that

$$\Pi'(a) = \left(\frac{1 + b_2}{1 - b_2}\right)^2 \frac{d}{\bar{\pi}_o(d)} e^{2\gamma_o \bar{\tau}_o} \frac{c}{\pi_+(c)} e^{2\gamma_+ \tau_+} \left(\frac{1 - b_2}{1 + b_2}\right)^2 \frac{b}{\pi_o(b)} e^{2\gamma_o \tau_o} = \frac{a}{\Pi(a)} e^{2(\gamma_o(\tau_o + \bar{\tau}_o) + \gamma_+ \tau_+)}, \tag{4}$$

with $\tau_+ \in (0, \pi)$ increasing with a, and $\bar{\tau}_o + \tau_o \in (0, 2\pi - \tau^*)$ decreasing with a. So $\gamma_o(\tau_o + \bar{\tau}_o) + \gamma_+ \tau_+$ is a decreasing function in a and $\gamma_o(\tau_o + \bar{\tau}_o) + \gamma_+ \tau_+ \to \gamma_+ \pi < 0$ when $a \to \infty$. Hence from (4) and from the fact that $\Pi'(a_s) \le 1$ we must have

$$\gamma_o(\tau_{os} + \bar{\tau}_{os}) + \gamma_+ \tau_{+s} \le 0.$$

Now supposing that Π admits a second fixed point a_r from the monotonicity of the function $\gamma_o(\tau_o + \bar{\tau}_o) + \gamma_+ \tau_+$ we must have

$$\gamma_o(\tau_{os} + \bar{\tau}_{os}) + \gamma_+ \tau_{+s} = 0 \text{ and } \gamma_o(\tau_{or} + \bar{\tau}_{or}) + \gamma_+ \tau_{+r} < 0.$$

As a_s and a_r are fixed point of Π this implies from (4) that $\Pi'(a_s) = 1$ and $\Pi'(a_r) < 1$. So in this case, for $a \in (a_s, a_r)$ we obtain $\Pi(a) > a$. Now from (4) it follows that $\Pi'(a) < 1$ for $a \in (a_s, a_r)$ and from the Mean Value Theorem we have

$$\Pi(a) - \Pi(a_s) = \Pi'(\bar{a})(a - a_s) < a - a_s$$

that implies that $\Pi(a) < a$. This is a contradiction and so we have at most a fixed point a_s for Π and

$$\gamma_o(\tau_{os} + \bar{\tau}_{os}) + \gamma_+ \tau_{+s} < 0. \tag{5}$$

On the other hand since

$$h'(a) = \Pi'(a) - 1 \text{ and } \lim_{a \to \infty} h'(a) = e^{2\gamma_+ \pi} - 1 < 0,$$

it follows by the Mean Value Theorem that $\lim_{a \to \infty} h(a) = -\infty$, and this shows that h admits a zero. Moreover this zero is equivalent to a fixed point of the first return map Π and this fixed point is a hyperbolic attractor because from (5) we have $\Pi'(a_s) < 1$.

Now suppose that $\gamma_o < 0$ and $\gamma_+ > 0$. In this case we can use the same idea of the previous case and we obtain a unique limit cycle that is a hyperbolic repeller.

Now from Proposition 15 of [4] it is not possible to have a limit cycle that visit only the regions R_o and R_+ otherwise we would have two hyperbolic attractor limit cycle containing the same equilibrium point what is not possible. This finish the prove of the result.

Proposition 7: Assume that system (3) satisfies assumptions $(\mathrm{H1})$ and $(\mathrm{H2})$. Suppose that the real equilibrium point in R_o is between L_o and L_+. Then there exists a unique limit cycle of (3), which is hyperbolic. This limit cycle visits the three regions R_-, R_o and R_+ if $D(\Pi) = [0, \infty)$ and $\Pi(0) > 0$ and visits only the regions R_o and R_+ otherwise.

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