Uniqueness of Limit Cycles For Liénard Differential Equations of Degree Four

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Salou, Spain, October 1-5, 2012

Congratulations To

Jaume Llibre

For His 60th Birthday !

Thanks For

His Mathematics

And

His Contributions To Mathematical Society

About Liénard Equations

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where $F(x) = \int_0^x f(x)dx$.

A basic problem is :

the system can have how many limit cycles ?

A lot of papers studied this problem for different $f(x)$ and $g(x)$.

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- The study of the number of limit cycles is important to understand the behavior of the system. The study is not easy in general.
- If $f(x)$ and $g(x)$ are polynomials, then the above basic problem is a restricted version of the Hilbert's 16th problem, which is open even for the quadratic case, although a lot of mathematicians have studied the Hilbert's 16th problem for more than 100 years.

Lins-de Melo-Pugh's conjecture

Consider a classical polynomial Liénard differential equation

$$\dot{x} = y - F(x),$$

$$\dot{y} = -x,$$

where $F(x)$ is a polynomial of degree n.

In 1977 A. Lins, W. de Melo and C. C. Pugh conjectured that the equation has

at most $\left[\frac{n-1}{2}\right]$ limit cycles,

where $\left[\frac{n-1}{2}\right]$ means the largest integer less than or equal to $\frac{n-1}{2}$.

Known Results

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- Not true for $n \geq 6$: at least 2 more limit cycles can appear.

  - P. De Maesschalck and F. Dumortier, 2011.

The last two results were obtained by using singular perturbations.

A partial Result on $n = 4$

Consider the Liénard differential equation of degree four

$$\dot{x} = y - F(x),$$

$$\dot{y} = -x,$$

where $F(x) = b_1 x + b_2 x^2 + b_3 x^3 + x^4$.

Xianwu Zeng proved in 1982 that if

$$b_1 < 0 < b_3, \quad b_3^3 - 4b_2 b_3 + 8b_1 \leq 0,$$

then the system has at most one limit cycle.

We will give a different proof for this result, and explain the geometric meaning of his conditions.

Main Result

The Lins-de Melo-Pugh conjecture is

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More precisely

Any classical Liénard differential equation of degree four has at most one limit cycle, and the limit cycle is hyperbolic, if it exists.

This result was published in JDE, 252 (2012), 3142-3162.

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- It is true or not for $n = 5$?

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Concerning the Lins-de Melo-Pugh conjecture

- It is true or not for $n = 5$?
- What is the maximal number of limit cycles for $n \geq 6$?

- P. De Maesschalck and F. Dumortier proved in 2011 that at least 2 more limit cycles can appear for $n \geq 6$. The next problem is

$\left[\frac{n-1}{2}\right] + 2$ is the maximal number of limit cycles for $n \geq 6$ or not ?

The Main Steps of the Proof for $n = 4$

- Put the equation to a ”normal form”;
- Find exact ranges of parameters for which the system may have limit cycles;
- Obtain the same sign of the divergence integral along any closed orbit;

  - Use some basic properties of orbits of Liénard equations;
  - Use symmetry-like property for partial estimetes;
  - Use different changes of variables for different cases;
  - Use the Green formula in some cases;
  - Use the Differential Inequality Theorem several times;
  - The exact range of parameters is crucial to check conditions;
  - $\cdots\ \cdots$

A Normal Form of the Equation

Without loss of generality, we can transform the equation to

$$\frac{dx}{dt} = y - F(x), \quad \frac{dy}{dt} = -(x - \lambda),$$

where λ is a constant, $F(x) = \frac{a}{2}x^2 + \frac{b}{3}x^3 + \frac{1}{4}x^4$, satisfying

$$a \geq 0, \quad b \geq 0, \quad a \geq \frac{2}{9}b^2.$$

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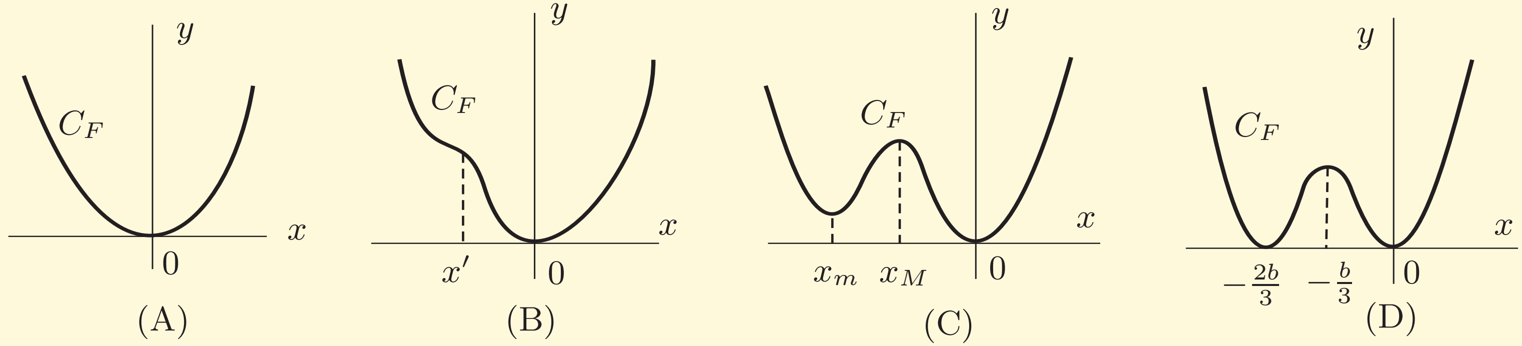
The shape of the curve $C_F := \{(x, y) : y = F(x)\}$ has 4 cases:

(A): $a > \frac{1}{4}b^2$, C_F has a unique minimum point;

(B): $a = \frac{1}{4}b^2$, C_F has a minimum and a inflection points;

(C): $\frac{2}{9}b^2 < a < \frac{1}{4}b^2$, C_F has two minimum and one maximum points;

(D): $a = \frac{2}{9}b^2$, a symmetry case of (C).



$$\frac{dx}{dt} = y - F(x), \quad \frac{dy}{dt} = -(x - \lambda),$$

The system may have limit cycles only in the case $a \geq 2b^2/9 > 0$ and

(I) $\lambda \in (-\frac{b}{3}, 0)$ in cases (A), (B), (C) and (D) $\Rightarrow$ the unique limit cycle is stable.

or

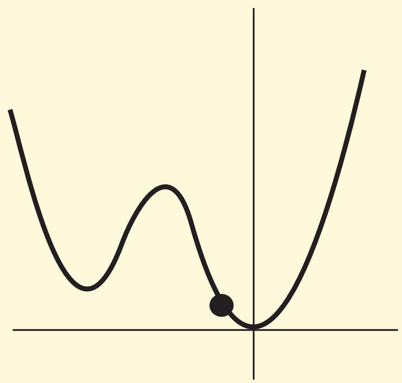
(II) $\lambda \in (x_m, x_M)$ in cases (C) and (D) $\Rightarrow$ the unique limit cycle is unstable.

This gives a range of the location of the unique singularity $(\lambda, F(\lambda))$.

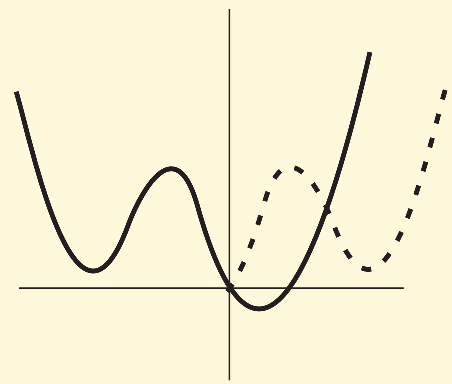
The condition of Zeng's result is exactly the case (I) above.

$$\frac{dx}{dt} = y - F(x), \quad \frac{dy}{dt} = -(x - \lambda) \qquad \Rightarrow \qquad \frac{dx}{dt} = y - E(x), \quad \frac{dy}{dt} = -x$$

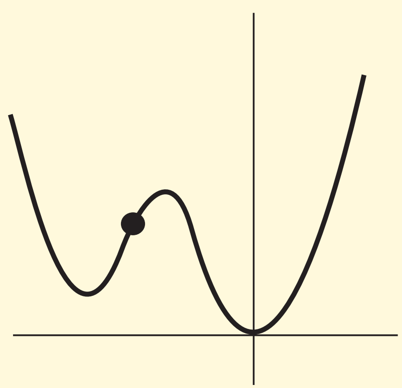
(I)



$\Rightarrow$

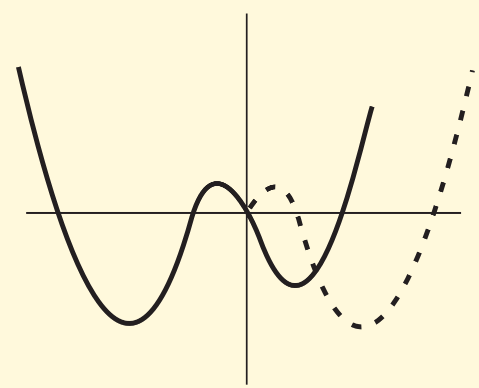


(II)



$\Rightarrow$

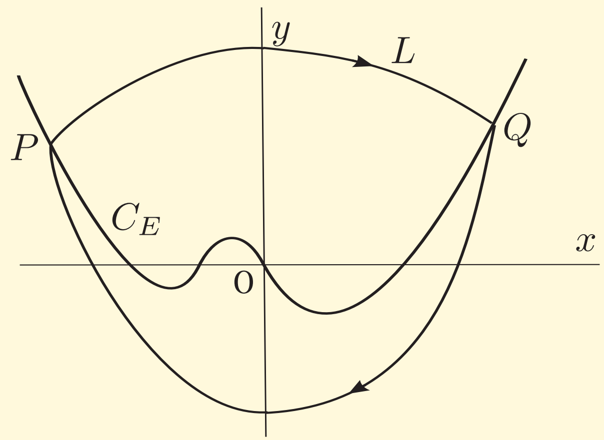
$x \mapsto -x$



Divergence Integral Along a Closed Orbit of Liénard Equation

$$\frac{dx}{dt} = y - E(x), \quad \frac{dy}{dt} = -x.$$

Suppose that the system has a closed orbit L, we consider



$$I_E(L) := -\oint_{L^+} E'(x)\, dt = \oint_{L^+} \frac{E'(x)}{x} dy = \oint_{L^+} \frac{E'(x)}{E(x) - y} dx,$$

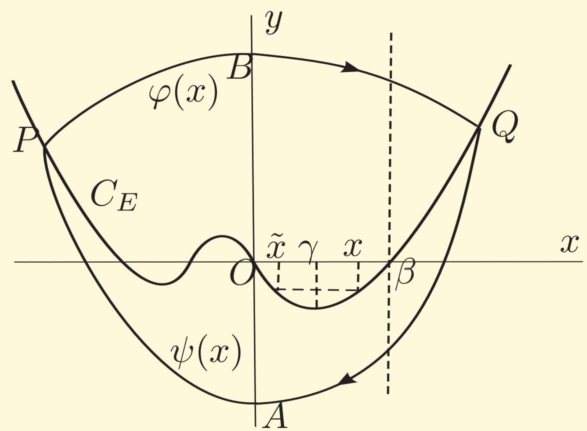
where L^+ means the integral is taken along L clockwise, given by the direction of the vector field. The different forms will be used in different places.

- If $I_E(L) < 0$ (> 0), then the orbit L is hyperbolic and stable (unstable).
- If $I_E(L)$ has the same sign for any orbit (if exists), then the system has at most one limit cycle.

Some Properties of Liénard Equations

$$\frac{dx}{dt} = y - E(x), \quad \frac{dy}{dt} = -x.$$

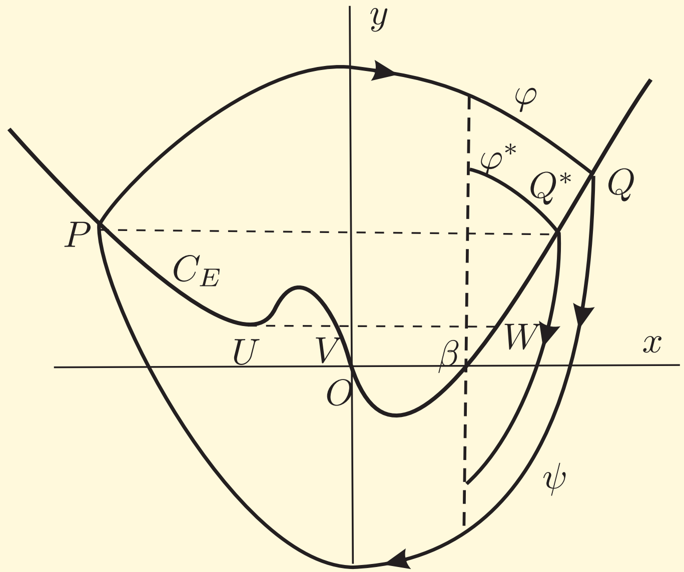
$E(x) < 0 < E(-x)$ for $0 < x \ll 1$ and $E(x) < E(-x)$ for $0 < x < x_0$, $E(x_0) = E(-x_0)$.



- $\psi(-x) < \psi(x) < 0 < \varphi(-x) < \varphi(x)$ for $0 < x < x_0$.
- $x_P < -x_0$ and $x_Q > x_0$.
- $y_P = F(x_P) < y_Q = F(x_Q)$.
- In region $-\lambda \leq x < +\infty$ the system has at most one limit cycle.
- Some symmetry-like property in the region with "U-arc" for $x > 0$ or "Ω-arc" for $x < 0$. We have $I_E[0, \beta] < 0$.

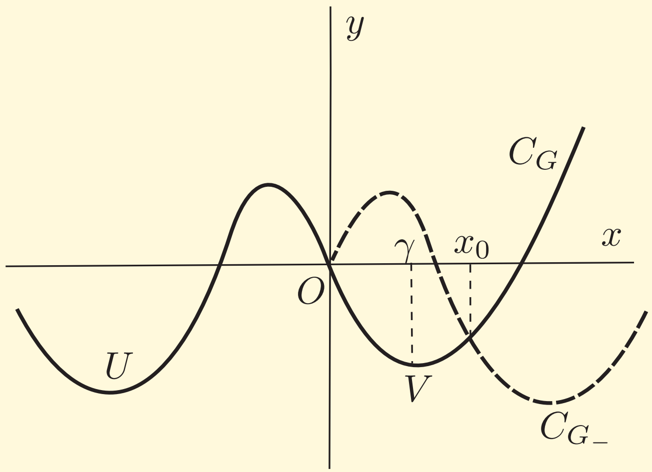
[By using Zhang Zhifen's Uniqueness Theorem, a result by Rychkov.
See the book by Dumortier, Llibre and Artés, for example]

An Example of Case (I)

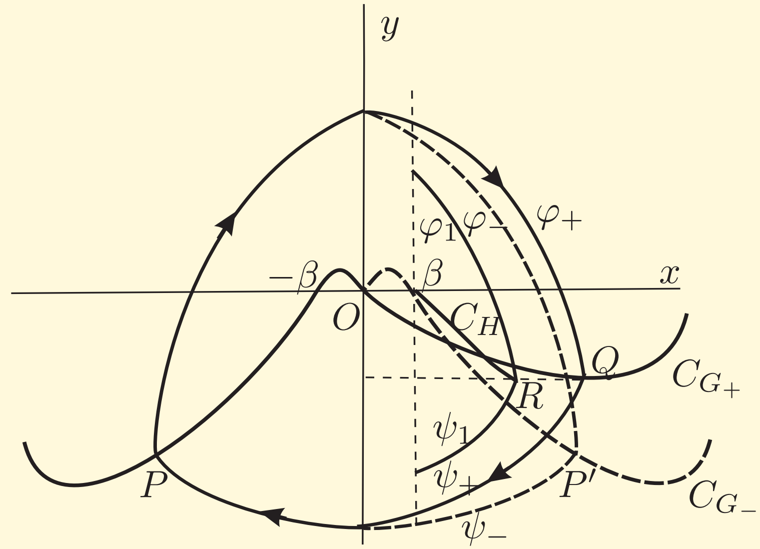


- By using the symmetry-like property: $I_E[x_U, x_V] + I_E[0, \beta] < 0$;
- By a similar estimate: $I_E[x_V, 0] + I_E[\beta, x_W] < 0$;
- By using the Green formula, a change of variables and the Differential Inequality Theorem
  $I_E[x_W, x_Q] < I_E^*[x_W, x_{Q^*}] < -I_E[x_P, x_U]$, i. e. $I_E[x_P, x_U] + I_E[x_W, x_Q] < 0$.

The Study of Case (II): x_P is right to left minimum



(i)



(ii)

Let $G(x) = E(-x)$. If $x_0 \geq \gamma$, or $x_0 < \gamma$ and $y_Q \geq 0$ the study is relatively simple. If $x_0 < \gamma$ and $y_Q < 0$, let $k = \frac{y_P}{y_Q} > 1$, $\bar{x} = k^2(x - \beta) + \beta \in (\beta, x_{P'})$ for $x \in (\beta, x_R)$;
Let $H(x) = \frac{1}{k}G(-\bar{x}(x))$, $\varphi_1(x) = \frac{1}{k}\varphi(-\bar{x}(x))$, $\psi_1(x) = \frac{1}{k}\psi(-\bar{x}(x))$. Then

$$I_G[x_P, -\beta] = \int_{x_P}^{-\beta} \left[\frac{G'(t)}{G(t) - \varphi(t)} - \frac{G'(t)}{G(t) - \psi(t)}\right] dt = \int_{\beta}^{x_R} \left[-\frac{H'(x)}{H(x) - \varphi_1(x)} + \frac{H'(x)}{H(x) - \psi_1(x)}\right] dx.$$

Define $\hat{x} = \hat{x}(x) \in (0, x_Q)$ by $H(x) = G(\hat{x}(x))$ for $x \in (\beta, x_R)$, $\varphi_2(x) = \varphi(\hat{x}(x)), \psi_2(x) = \psi(\hat{x}(x))$. Then

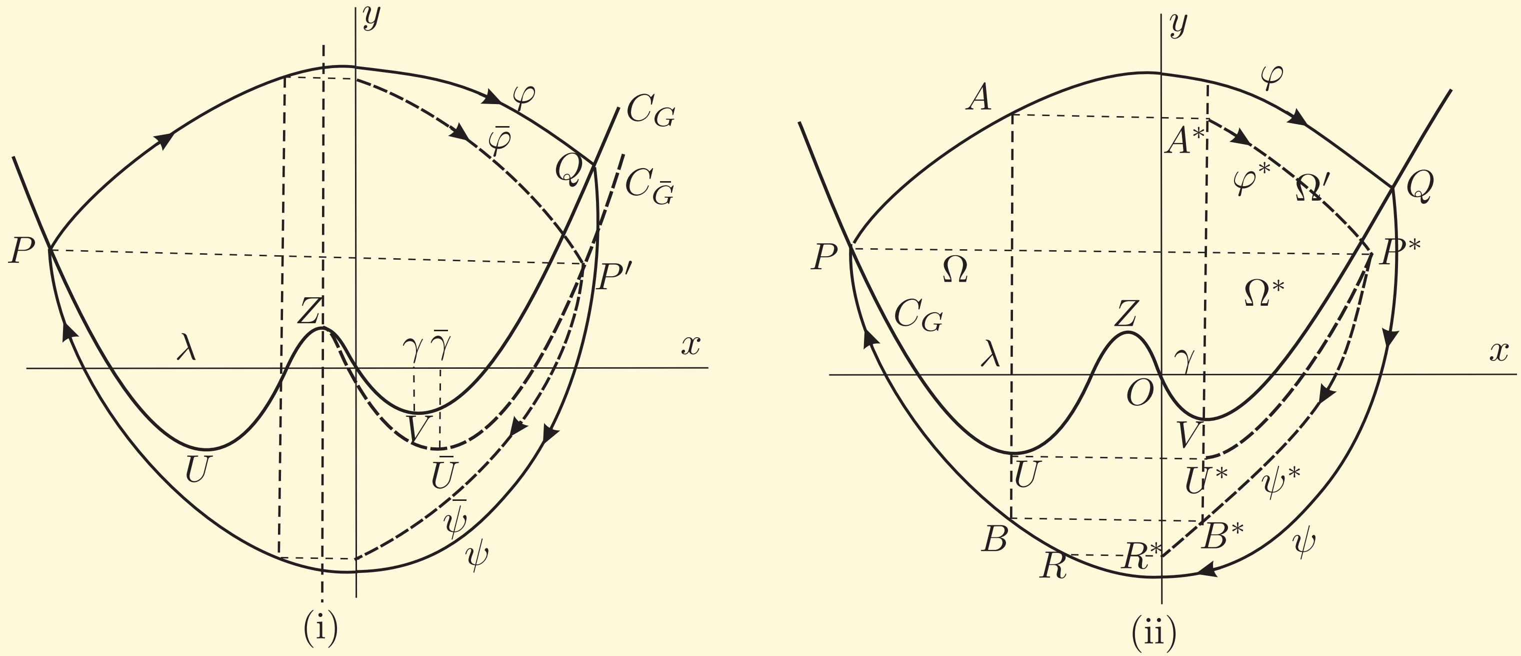
$$I_G[0, x_Q] = \int_{0}^{x_Q} \left[\frac{G'(t)}{G(t) - \varphi(t)} - \frac{G'(t)}{G(t) - \psi(t)}\right] dt = \int_{\beta}^{x_R} \left[-\frac{H'(x)}{H(x) - \varphi_2(x)} + \frac{H'(x)}{H(x) - \psi_2(x)}\right] dx.$$

$$I_G[X_P, -\beta] + I_G[0, x_Q] = \int_{\beta}^{x_R} \left[\frac{H'(x)(\varphi_2(x) - \varphi_1(x))}{(H(x) - \varphi_1(x))(H(x) - \varphi_2(x))} + \frac{H'(x)(\psi_1(x) - \psi_2(x))}{(H(x) - \psi_1(x))(H(x) - \psi_2(x))}\right] dx.$$

Some Facts need to be Proved

- $H'(x) < 0$ for $x \in (\beta, x_R)$.
- $C_H \cap C_{G_+}$ consists of a unique point.
- $x_R < x_Q$ and $y_R = y_Q$.
- $\varphi_2(x) - \varphi_1(x) > 0, \quad \psi_1(x) - \psi_2(x) > 0$ for $x \in (\beta, x_R)$.

The Study of Case (II): x_P is left to left minimum



The two minimum points at U and V, $x_U = \lambda < 0, x_V = \gamma > 0$. We prove 3 things:

(1) $x_Q > \gamma$ by using Differential Inequality Theorem $x_Q > x_{P'} > \bar{\gamma} > \gamma$.

(2) $I_G[\lambda, \gamma] < 0$ by using the same methods and transformations as above.

(3) $I_G[x_P, \lambda] + I_G[\gamma, x_Q] < 0$ by using Differential Inequality Theorem and Green formula

Some Computations in the Last Step

$$\iint_{\Omega^*}\left[\frac{d}{dx}\left(\frac{G'(x)}{x}\right)-\frac{d}{dx}\left(\frac{G'(-(x+c))}{-(x+c)}\right)\right]dxdy+\iint_{\Omega'\setminus\Omega^*}\frac{d}{dx}\left(\frac{G'(x)}{x}\right)dxdy,$$

where $c=|\lambda|-\gamma=x_m-2\lambda>0$. The first integrand is equal to

$$-2(b+3\lambda+c)+\lambda(\lambda^2+b\lambda+a)\left(\frac{1}{x^2}+\frac{1}{(x+c)^2}\right)>0,$$

because $x\geq\gamma>0$, $b+3\lambda+c=b+x_m+\lambda<b+x_m+x_M=0$, since

$\lambda_m<\lambda_M$ are the two negative roots of $\lambda^2+b\lambda+a=0$ in this case;

hence we also have $\lambda(\lambda^2+b\lambda+a)>0$ for $\lambda<0$ and $\lambda\in(x_m,x_M)$.

The second integrand is also positive, because

$$\frac{d}{dx}\left(\frac{G'(x)}{x}\right)=\frac{1}{x^2}\left[2x^3-(b+3\lambda)x^2+\lambda(\lambda^2+b\lambda+a)\right]>0,$$

since $x>0$ and $b+3\lambda<0$.

THANK YOU VERY MUCH!