Topological and algebraic reducibility for patterns on trees

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Patterns

$f : X \to X$ continuous, P a finite f-invariant subset of X ($f(P)=P$).

The *pattern of P* is a combinatorial object that contains all the information about (at least):

- The relative positions of the points of P inside the space X
- The way these positions are permuted under the action of f

Entropy of a pattern

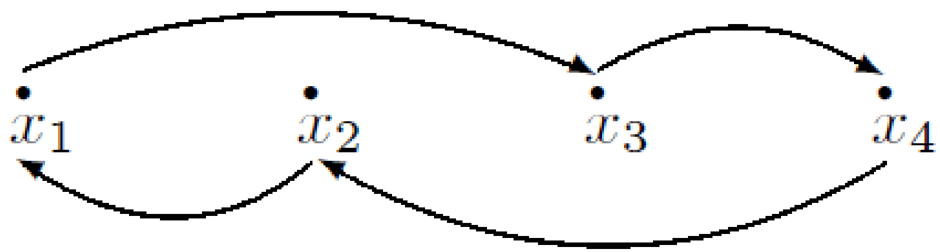
$f : X \to X$ continuous, P a finite f-invariant subset of X. Let θ be the pattern of P.

The <u>*entropy*</u> of θ, $h(\theta)$, is the infimum of the topological entropies of all self-maps of X exhibiting an invariant set with pattern θ:

$$h(\theta) = \inf \{ h(g) \text{ s.t. } g : X \to X \text{ exhibits } \theta \}$$

Interval patterns

- When *X* is an interval, the *pattern* of *P* can be identified with a permutation in the natural way: $\theta = (3,1,4,2)$



Interval patterns – II

The entropy of an interval pattern θ can be easily computed using some algebraic tools:

Piecewise monotone (or "connect-the-dots") map $\rightarrow$ Markov matrix $\rightarrow$ $h(\theta)$ = logarithm of its spectral radius

Other 1-dimensional spaces

- Ll. Alsedà, J. Guaschi, J. Los, F. Mañosas, P. Mumbrú, *Canonical representatives for patterns of tree maps*, Topology 36 (1997).

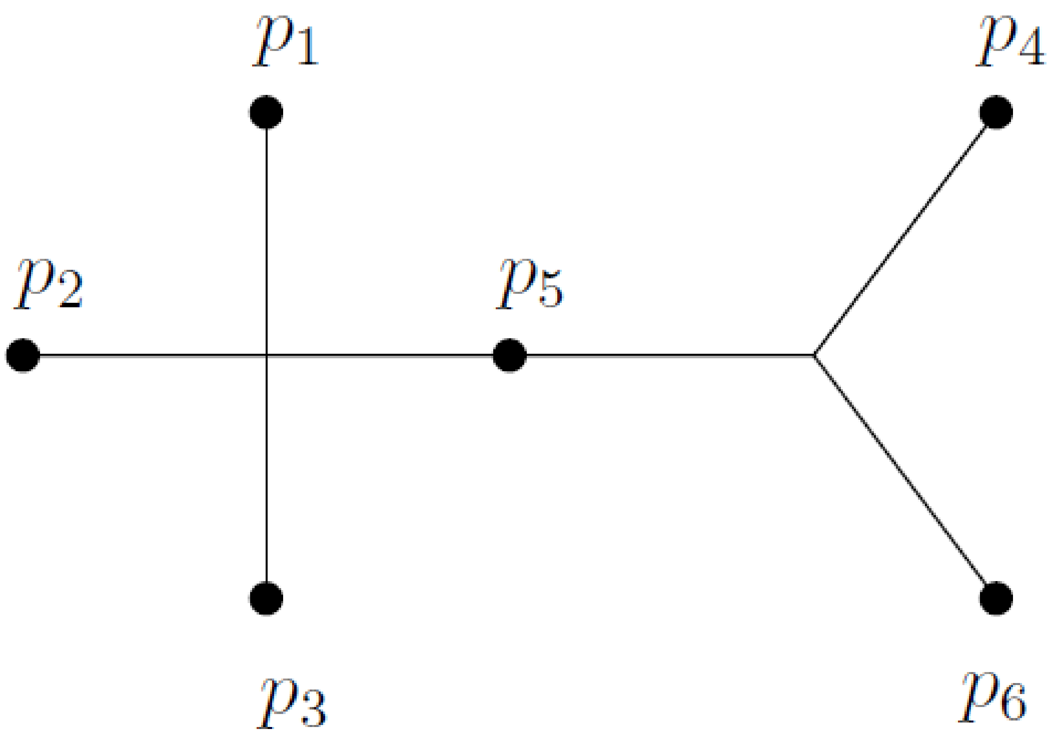
- Ll. Alsedà, F. Gautero, J. Guaschi, J. Los, F. Mañosas, P. Mumbrú, *Patterns and minimal dynamics for graph maps*, Proc. London Math. Soc. (3) 91 (2005).

Tree patterns

Some terminology:

- A *model* is a triplet (T,P,f), where T is a tree, $f: T \rightarrow T$ is continuous and P is a finite f-invariant subset of T.
- Two points x,y of P are *consecutive* if (x,y) contains no points of P.
- Any maximal subset of P consisting only of pairwise consecutive points is a *discrete component*.

Tree patterns - II



(T, P)

Tree patterns – III

In this context (tree maps), a *pattern* is an abstract object which can be identified with the conjugacy class of all models with a fixed distribution of discrete components and images of points in the distinguished finite invariant set.

Tree patterns -IV

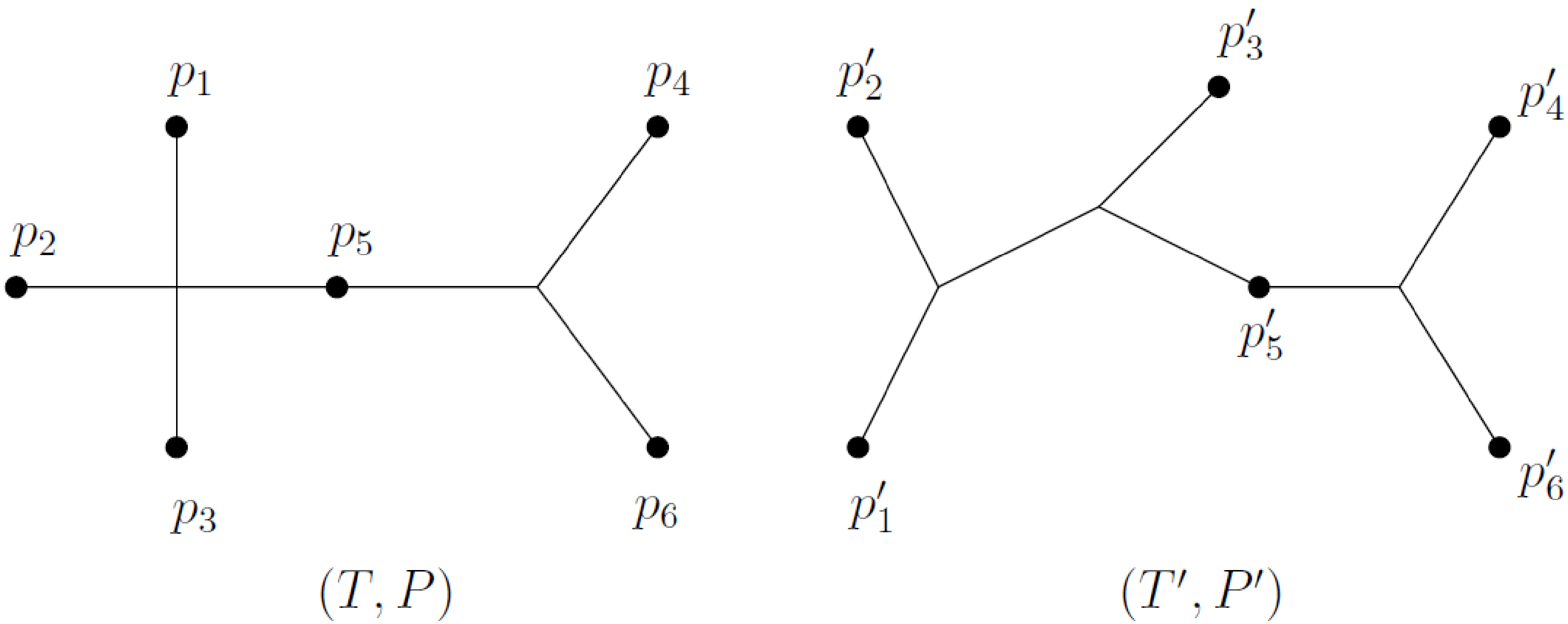


FIGURE 1. Set $P = \{p_i\}_{i=1}^{6}$ and $P' = \{p'_i\}_{i=1}^{6}$. If $f\colon T \longrightarrow T$ and $f'\colon T' \longrightarrow T'$ are tree maps such that $f(p_i) = p_{i+1}$ and $f'(p'_i) = p'_{p+1}$ for $1 \leq i \leq 5$, $f(p_6) = p_1$ and $f'(p'_6) = p'_1$, then the models (T, P, f) and (T', P', f') represent the same cyclic pattern π.

Tree patterns – V

- Given a tree pattern θ (which is an equivalence class of models), one can construct a monotone representative *(T,P,f)* with properties very similar to those of the “connect-the-dots” interval maps:

Monotone model $\rightarrow$ Markov matrix $\rightarrow$ $h(\theta)$ = logarithm of its spectral radius

Tree patterns – VI

Theorem: Given any tree pattern θ, there exists a model (called <u>*canonical model*</u>) (T,P,f) such that f exhibits the pattern θ over P and f is <u>monotone</u> on any interval $[a,b]$ defined by a pair a,b of consecutive points of P. Moreover, $h(f) = h(\theta)$.

[Example: blackboard]

The canonical model is essentially <u>unique</u>.

Tree patterns – VII

Given a pattern θ and its canonical model *(T,P,f)*, the *f*-image of each vertex of *T* is uniquely determined and is either a vertex or belongs to *P*. Thus, *P* U V(*T*) is an invariant set. [Example: blackboard]

- So, P U V(T) defines a Markov partition (a set of maximal closed intervals whose interiors do not intersect P U V(T)).

Tree patterns – VIII

- Canonical model → Markov partition of k intervals → k x k Markov matrix M → $h(\theta)$ = logarithm of its spectral radius

- M is a binary matrix: $M_{ij} = 1$ if the i-th interval f-covers the j-th one, 0 otherwise.

A remark: collapsing intervals

- Let (T,P,f) be the canonical model of a pattern. It may happen that, for some Markov interval [a,b], f(a)=f(b), implying that the f-image of [a,b] collapses to one point: [a,b] is called a *collapsing interval*.

[Example: blackboard]

Reducibility

- Reducible systems are those such that the space can be decomposed in connected pieces with pairwise disjoint interiors which are permuted by the map.
- In this situation the behaviour of the original map can be related with the dynamics of an iterate of the map on the reduced pieces.

Aim

- We want to clarify in full the notion of reducibility (irreducibility) for periodic tree patterns. Next we provide precise definitions of these notions and study their dynamical implications at a topological level.
- We also relate these features to the algebraic properties of the Markov matrix.

Some Algebra

Recall that a nonnegative matrix M is called *reducible* if, for a permutation matrix A,

$$A^T M A = \begin{pmatrix} M_{11} & 0 \\ M_{21} & M_{22} \end{pmatrix}$$

where M_{11} and M_{22} are square matrices.

Otherwise M is called *irreducible*.

Some Algebra- II

- An irreducible matrix M is called *primitive* if all powers M^n are irreducible for $n \geq 2$. Otherwise M is called *imprimitive*.

- It is well known (Perron-Frobenius) that an irreducible matrix M is primitive if and only if there exists $n \geq 1$ such that all the entries of M^n are positive.

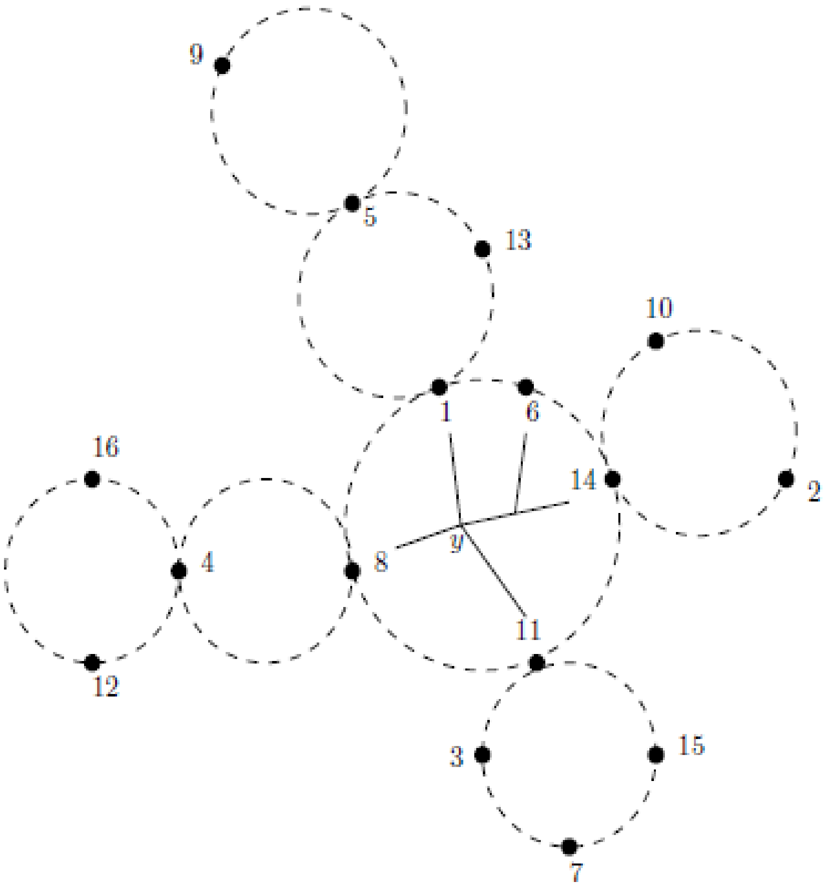
Block structures

- Let T be a tree. For any $X \subset T$, the convex hull $\langle X \rangle$ of X is the smallest connected subset of T containing X.
- Let (T,P,f) be the canonical model of an n-cyclic pattern. The pattern is said to have a *p-<u>block structure</u>* if there is a partition $P = P_1 \cup P_2 \cup . . . \cup P_p$ with $f(P_i) = P_{i+1}$ and $\langle P_i \rangle \cap P_j = \emptyset$ for any pair $\{ i, j \}$.

Block structures - II

- If in addition $\langle P_i \rangle \cap \langle P_j \rangle = \emptyset$ for any pair then the block structure is called *separated*.

Block structures - III



{1,3,5,7,9,11,13,15} ∪ {2,4,6,8,10,12,14,16} is a (non separated) 2-block structure.

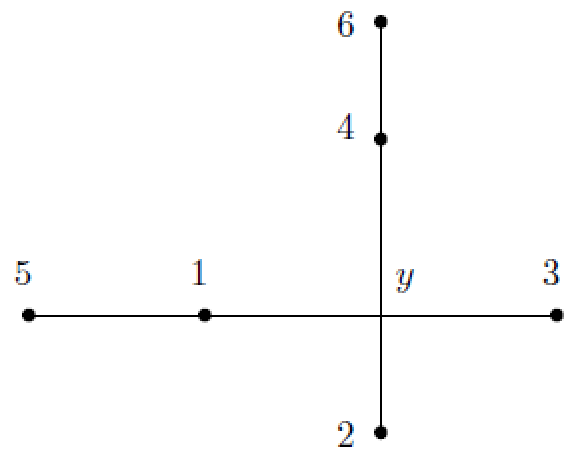
{1,5,9,13} ∪ {2,6,10,14} ∪ {3,7,11,15} ∪ {4,8,12,16} is a separated) 4-block structure.

Some results

Let θ be a cyclic pattern and let M be its associated Markov matrix.

- **Theorem 1**: M is reducible if and only if θ has either separated block structures or collapsing intervals.
- **Theorem 2**: M is primitive if and only if θ has no block structures and no collapsing intervals.

An example



θ has a unique block-structure, {1,3,5} ∪ {2,4,6}, which is not separated. By Theorem 1, M is irreducible. Moreover, by Theorem 2, M is imprimitive.

An example - II

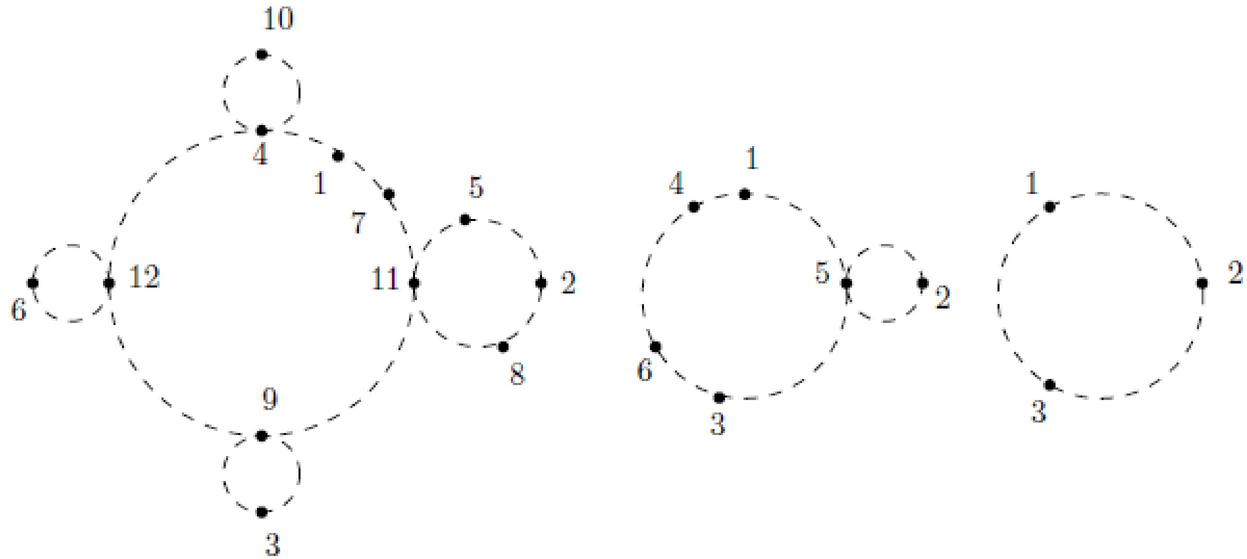
- This is in contrast with the interval case! For interval patterns, if the Markov matrix is irreducible, then it is primitive:

- **Theorem 3**: Let θ be a *simplicial* cyclic pattern and let *M* be its Markov matrix. If *M* is irreducible then *M* is primitive.

The skeleton

- When a cyclic pattern has a *p*-block structure one can construct its *skeleton*, which is a *p*-cyclic pattern obtained by collapsing each block to one point and defining a *p*-periodic map in the obvious way.

The skeleton – an example



- A sequence of skeletons:

{1,7} ∪ {2,8} ∪ {3,9} ∪ {4,10} ∪ {5,11} ∪ {6,12} →

→ {1,4} ∪ {2,5} ∪ {3,6} →

→ trivial 3-cyclic pattern (one discrete component)

Starry patterns

- A cyclic pattern θ is called <u>*k-starry*</u> if there is a sequence of patterns θ_1, θ_2, …, θ_k such that $\theta_1 = \theta$, θ_{i+1} is the skeleton of θ_i and θ_k is a trivial pattern (one discrete component).

- An example: the 12-cyclic pattern in the previous slide.

Starry patterns - II

- **Theorem 4**: The entropy of a cyclic pattern θ is zero if and only θ is k-starry for some k.

Starry patterns - III

Surprisingly, the following results were unknown even for the interval case:

- **Corollary 5**: Let be an *n*-cyclic pattern $\theta = [\ T,P,f\ ]$. Then, $h(\theta)=0$ if and only if all patterns $[\ T,P,f^k\ ]$, with $\gcd(k,n)=1$, have entropy zero.

Starry patterns - IV

- **Corollary 6**: Let be an n-cyclic pattern $\theta = [\ T,P,f\ ]$. Then, $h(\theta)>0$ if and only if all patterns $[\ T,P,f^k\ ]$, with $\gcd(k,n)=1$, have positive entropy.