On the dynamics of the rigid body with a fixed point: periodic orbits and integrability

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1. Introduction and statement of the main results

The aim of the present contribution is to study the periodic orbits of a rigid body with a fixed point and quasi–spherical shape under the effect of a Newtonian force field given by different small potentials. For studying these periodic orbits we shall use averaging theory. Moreover, we provide information on the $\mathcal{C}^1$–integrability of these motions.

The motion of a rigid body with a fixed point is described by the Hamiltonian equations associated to the Hamiltonian

$$\mathcal{H} = \frac{(G^2-L^2)}{2}\left(\frac{\sin^2 l}{A}+\frac{\cos^2 l}{B}\right)+\frac{L^2}{C}+U(k_1,k_2,k_3) \tag{1}$$

with

$$k_1=\left(\frac{H}{G}\sqrt{1-\left(\frac{L}{G}\right)^2}+\frac{L}{G}\sqrt{1-\left(\frac{H}{G}\right)^2}\cos g\right)\sin l+\sqrt{1-\left(\frac{H}{G}\right)^2}\sin g\cos l,$$

$$k_2=\left(\frac{H}{G}\sqrt{1-\left(\frac{L}{G}\right)^2}+\frac{L}{G}\sqrt{1-\left(\frac{H}{G}\right)^2}\cos g\right)\cos l-\sqrt{1-\left(\frac{H}{G}\right)^2}\sin g\sin l,$$

$$k_3=\left(\frac{H}{G}\right)\left(\frac{L}{G}\right)-\sqrt{1-\left(\frac{L}{G}\right)^2}\sqrt{1-\left(\frac{H}{G}\right)^2}\cos g.$$

This is a Hamiltonian in the Andoyer–Deprit canonical variables (L,G,l,g) of two degree of freedom with the positive parameters A, B, C and H.

We introduced the parameters $\alpha = \dfrac{\frac{1}{A}+\frac{1}{B}-\frac{2}{C}}{\frac{2}{C}}$, $\beta = \dfrac{\frac{1}{A}-\frac{1}{B}}{\frac{1}{A}+\frac{1}{B}-\frac{2}{C}}$. The parameter β is known as the triaxial coefficient. Note that α can take any positive value depending on the physical characteristics of the rigid body. But the triaxial coefficient β is bounded between zero (the oblate spheroid $A=B$) and one (the prolate spheroid $B=C$), although it is undefined in the limit case of a sphere, taking any value between zero and one depending on the direction in which we approach the limit. See for more details [1].

In this work we assume that $0<\alpha=\varepsilon^k \ll 1$, i.e. Then the Hamiltonian (1) is expressed by

$$\mathcal{H} = \frac{G^2}{2C}+\varepsilon^k\mathcal{P}_1+U(k_1,k_2,k_3),$$

where $\mathcal{P}_1=\frac{1}{2C}\left(G^2-L^2\right)(1-\beta\cos 2l)$. Moreover we shall consider the following three cases:

Case 1: $U(k_1,k_2,k_3)=\varepsilon V(k_1,k_2,k_3)$ and $k=2$, i.e.

$$\mathcal{H} = \frac{G^2}{2C}+\varepsilon\mathcal{P}_2+\varepsilon^2\mathcal{P}_1. \tag{2}$$

where $\mathcal{P}_2=V(k_1,k_2,k_3)$.

Case 2: $U(k_1,k_2,k_3)=\varepsilon V(k_1,k_2,k_3)$ and $k=1$, i.e.

$$\mathcal{H} = \frac{G^2}{2C}+\varepsilon(\mathcal{P}_1+\mathcal{P}_2). \tag{3}$$

Case 3: $U(k_1,k_2,k_3)=\varepsilon^2 V(k_1,k_2,k_3)$ and $k=1$, i.e.

$$\mathcal{H} = \frac{G^2}{2C}+\varepsilon\mathcal{P}_2+\varepsilon^2\mathcal{P}_1. \tag{4}$$

We note that $\mathcal{P}_1$ measures the difference of the shape of the rigid body between a sphere and a tri-axial ellipsoid, and $\mathcal{P}_2$ measures the external forces acting on the rigid body. We shall assume that the perturbing functions $\mathcal{P}_i$ are smooth in the variables $(L,l;G,g)$.

Theorem 1

We consider the motion of the rigid body with a fixed point under the action of the Hamiltonian (2). On the energy level $\mathcal{H}=h>0$ if $\varepsilon\neq 0$ is sufficiently small, then for every zero (L_0,l_0) of the system

$$f_1^1(L,l)=-\frac{1}{2\pi}\sqrt{\frac{C}{2h}}\int_0^{2\pi}\frac{\partial\mathcal{P}_2}{\partial l}dg=0,$$

$$f_1^2(L,l)=\frac{1}{2\pi}\sqrt{\frac{C}{2h}}\int_0^{2\pi}\frac{\partial\mathcal{P}_2}{\partial L}dg=0,$$

satisfying that

$$\det\left(\left.\frac{\partial(f_1^1,f_1^2)}{\partial(L,l)}\right|_{(L,l)=(L_0,l_0)}\right)\neq 0, \tag{5}$$

there exists a $2\pi-$periodic solution $(L(g,\varepsilon),l(g,\varepsilon),G(g,\varepsilon))$ in the variable g of the rigid body such that $(L(g,0),l(g,0),G(g,0))=(L_0,l_0,\sqrt{2Ch})$ when $\varepsilon\to 0$.

Let $R=(a^2-b^2)^2(a^2+b^2)+(a^4-6a^2b^2+b^4)c^2$. An application of Theorem 1 is Corollary 2. It describes the motion of a non–homogeneous sphere with center of mass at the point (a,b,c) under a weak gravitational Newtonian potential.

Corollary 2

A spherical rigid body with Hamiltonian (2), weak potential $\mathcal{P}_2=ak_1+bk_2+ck_3$ with a, b and c positive and $\varepsilon\neq 0$ sufficiently small has in every positive energy level at least four linear stable periodic orbits if $R>0$, two linear stable periodic orbits if $R=0$, and two linear stable periodic orbits and two unstable ones if $R<0$.

Theorem 3

We consider the motion of the rigid body with a fixed point under the action of the Hamiltonian (3). On the energy level $\mathcal{H}=h>0$ if $\varepsilon\neq 0$ is sufficiently small, then for every zero (L_0,l_0) of the system

$$f_1^1(L,l)= -\frac{1}{2\pi}\sqrt{\frac{C}{2h}}\int_0^{2\pi}\frac{\partial(\mathcal{P}_1+\mathcal{P}_2)}{\partial l}dg=0,$$

$$f_1^2(L,l)= \frac{1}{2\pi}\sqrt{\frac{C}{2h}}\int_0^{2\pi}\frac{\partial(\mathcal{P}_1+\mathcal{P}_2)}{\partial L}dg=0,$$

satisfying (5), there exists a $2\pi-$periodic solution $(L(g,\varepsilon),l(g,\varepsilon),G(g,\varepsilon))$ in the variable g of the rigid body such that $(L(g,0),l(g,0),G(g,0))=(L_0,l_0,\sqrt{2Ch})$ when $\varepsilon\to 0$.

Corollary 4

A quasi–spherical rigid body with Hamiltonian (3), weak potential $\mathcal{P}_2=ck_3$ with $c>0$ and $\varepsilon\neq 0$ sufficiently small can have at least eight periodic orbits in every positive energy level.

Corollary 4 describes the motion of a non–homogeneous quasi–spherical rigid body with center of mass at the point $(0,0,c)$ under a weak gravitational Newtonian potential. The linear stability of the periodic orbits described in Corollary 4 can be studied using the averaging theory as well.

Theorem 5

We consider the motion of the rigid body with a fixed point under the action of the Hamiltonian (4). On the energy level $\mathcal{H}=h>0$ if $\varepsilon\neq 0$ is sufficiently small and

$$\int_0^{2\pi}\frac{\partial\mathcal{P}_1}{\partial l}=\int_0^{2\pi}\frac{\partial\mathcal{P}_1}{\partial L}=0, \tag{6}$$

then for every zero (L_0,l_0) of the system

$$f_2^1(L,l)= \frac{1}{2\pi}\int_0^{2\pi}F_2^1(L,l,g)dg=0,$$

$$f_2^2(L,l)= \frac{1}{2\pi}\int_0^{2\pi}F_2^2(L,l,g)dg=0,$$

where

$$F_2^1=\frac{1}{4h^2}\left(2hC\left(-\frac{\partial^2\mathcal{P}_1}{\partial l^2}\left(\int_0^g\frac{\partial\mathcal{P}_1}{\partial L}dg\right)+\right.\right.$$
$$\left.\left.+\frac{\partial^2\mathcal{P}_1}{\partial L\partial l}\left(\int_0^g\frac{\partial\mathcal{P}_1}{\partial l}dg\right)\right)+\sqrt{\frac{C}{(2h)^3}}\mathcal{P}_1\right.$$
$$\left.\left(-\frac{\partial\mathcal{P}_1}{\partial l}+\sqrt{2Ch}\frac{\partial^2\mathcal{P}_1}{\partial l\partial G}\right)+\frac{C}{2h}\frac{\partial\mathcal{P}_1}{\partial G}\frac{\partial\mathcal{P}_1}{\partial l}-\sqrt{\frac{C}{2h}}\frac{\partial\mathcal{P}_2}{\partial l}\right)$$

$$F_2^2=\frac{1}{4h^2}\left(2hC\left(-\frac{\partial^2\mathcal{P}_1}{\partial L^2}\left(\int_0^g\frac{\partial\mathcal{P}_1}{\partial l}dg\right)+\right.\right.$$
$$\left.\left.++\frac{\partial^2\mathcal{P}_1}{\partial L\partial l}\left(\int_0^g\frac{\partial\mathcal{P}_1}{\partial L}dg\right)\right)+\sqrt{\frac{C}{(2h)^3}}\mathcal{P}_1\right.$$
$$\left.\left(\frac{\partial\mathcal{P}_1}{\partial L}-\sqrt{2Ch}\frac{\partial^2\mathcal{P}_1}{\partial L\partial G}\right)-\frac{C}{2h}\frac{\partial\mathcal{P}_1}{\partial G}\frac{\partial\mathcal{P}_1}{\partial L}+\sqrt{\frac{C}{2h}}\frac{\partial\mathcal{P}_2}{\partial L}\right)$$

satisfying (5), there exists a $2\pi-$periodic solution $(L(g,\varepsilon),l(g,\varepsilon),G(g,\varepsilon))$ in the variable g of the rigid body such that $(L(g,0),l(g,0),G(g,0))=(L_0,l_0,\sqrt{2Ch})$ when $\varepsilon\to 0$.

Corollary 6

A quasi–spherical rigid body with Hamiltonian (4), weak potential $\mathcal{P}_2=ck_3^2$ with $c>0$, energy level $\mathcal{H}=h=3H^2/(2C)$ and $\varepsilon\neq 0$ sufficiently small can have at least fourteen periodic solutions.

In the proofs of Theorems 1 and 3 we shall use the averaging theory of first order, and in the proof of Theorem 5 we shall use the averaging theory of second order. The $\mathcal{C}^1$ non–integrability in the sense of Liouville–Arnold of this problem can be studied, see [2].

2. Proofs and details

To see the proofs and details see paper [2], which is currently submitted, at the web

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http://www.dmae.upct.es/~jlguirao
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3. Bibliography

[1] H. Kinoshita, *First order perturbations of the two finite body problem*, Publications of the Astronomical Society of Japan, **24**, (1972), 423–457.

[2] J.L.G. Guirao, J. Llibre, J.A. Vera, *On the dynamics of the rigid body with a fixed point: periodic orbits and integrability*, 2012, submitted.