Liouvillian integrability and invariant algebraic curves of ordinary differential equations

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Introduction

Consider a planar polynomial differential system

$$\frac{dx}{dt} = \dot{x} = P(x,y), \qquad \frac{dy}{dt} = \dot{y} = Q(x,y), \tag{1}$$

where the dependent variables x and y are complex, the independent one (the *time*) t is real or complex, and $P, Q \in \mathbb{C}[x,y]$, where $\mathbb{C}[x,y]$ is the ring of all polynomials in the variables x and y with coefficients in $\mathbb{C}$. We denote by $\mathrm{d} = \max\{\deg P, \deg Q\}$ the *degree* of the polynomial system.

There was the belief that a Liouvillian integrable system (see definitions below) has always an invariant algebraic curve. Moreover, this claim was proved adding certain hypotheses, see [6]. However this claim was refuted in [3] where it is proved that there exist Liouvillian integrable polynomial systems without any finite invariant algebraic curve.

The present poster is based upon the work [5] where we assume that system (1) is Liouvillian integrable and determine, in terms of the degree in y of $P(x,y)$ and $Q(x,y)$, when this implies that the equation has a finite invariant algebraic curve. Indeed, we provide examples in which, if the condition on the degree in y is not satisfied, then the system is Liouvillian integrable but has no finite invariant algebraic curve.

Definitions

- Let $f = f(x,y) = 0$ be an algebraic curve in $\mathbb{C}^2$. We say that it is a finite invariant algebraic curve by the polynomial system (1) if
$$P\frac{\partial f}{\partial x} + Q\frac{\partial f}{\partial y} = kf, \tag{2}$$
for some polynomial $k = k(x,y) \in \mathbb{C}[x,y]$, called the ***cofactor*** of the algebraic curve $f = 0$. Note that the degree of the polynomial k is at most $\mathrm{d}-1$. From (2) it is immediate to check that the algebraic curve $f = 0$ is formed by trajectories of the polynomial system (1).
- Let $h, g \in \mathbb{C}[x,y]$ and assume that h and g are relatively prime in the ring $\mathbb{C}[x,y]$. Then the function $\exp(g/h)$ is called an exponential factor of the polynomial system (1) if for some polynomial $k \in \mathbb{C}[x,y]$ of degree at most $\mathrm{d}-1$ it satisfies equation
$$P\frac{\partial \exp(g/h)}{\partial x} + Q\frac{\partial \exp(g/h)}{\partial y} = k\,\exp(g/h). \tag{3}$$
If $\exp(g/h)$ is an exponential factor it is easy to show that $h = 0$ is an invariant algebraic curve.
- Let U be a non-empty open subset of $\mathbb{C}^2$. We say that a not locally constant function $H : U \to \mathbb{C}$ is a first integral of the polynomial system (1) in U if H is constant on the trajectories of the polynomial system (1) contained in U.
- We say that a not locally null function $R : U \to \mathbb{C}$ is an integrating factor of the polynomial system (1) in U if R satisfies that
$$\frac{\partial(RP)}{\partial x} + \frac{\partial(RQ)}{\partial y} = 0,$$
in the points $(x,y) \in U$.
- We recall that, intuitively, a complex Liouvillian function is one that it is obtained from complex rational functions by a finite process of integrations, exponentiations and algebraic operations. We say that system (1) is Liouvillian integrable if it has a first integral of Liouvillian type.

Preliminary results

The following result relates the Liovillian integrability of a system (1) with the possible form of an integrating factor made of exponential factors and finite invariant algebraic curves.

In [1] C. Christopher showed that a polynomial system (1) has a Liouvillian first integral iff the system has an integrating factor of the form
$$R = \exp(g/h)\prod f_i^{\lambda_i}, \tag{4}$$
where g, h and f_i are polynomials and $\lambda_i \in \mathbb{C}$. Indeed, if $g \not\equiv 0$, $\exp(g/h)$ is an exponential factor of system (1) and $f_i = 0$ are finite invariant algebraic curves of system (1).

The next result provides the first example of a planar polynomial differential system with a Liouvillian first integral but no finite invariant algebraic curves.

In [3] it is proved that the quadratic system
$$\dot{x} = -1 - x(2x+y), \qquad \dot{y} = 2x(2x+y), \tag{5}$$
is Liouvillian integrable and has no finite invariant algebraic curves.

The particular case of systems of the form (1) with $P(x,y) = 1$ are studied in [4] where the following results are given.

In [4], J. Giné and J. Llibre consider planar polynomial differential systems of the form:
$$\dot{x} = 1, \qquad \dot{y} = a_0(x) + a_1(x)y + a_2(x)y^2 + \ldots + a_n(x)y^n, \tag{6}$$
with $a_i(x) \in \mathbb{C}[x]$ for $i = 1, 2, \ldots, n$, $a_n(x) \neq 0$.
The following statements hold:

- If $n \geq 2$ and system (6) is Liouvillian integrable, then it has a finite invariant algebraic curve.
- There are systems of the form (6) with $n = 1$ which are Liouvillian integrable and have no finite invariant algebraic curve. For example $\dot{x} = 1$, $\dot{y} = (1+xy)/2$.

Main results

Our aim in [5] is to generalize the previous results of [3, 4] to an arbitrary planar polynomial differential systems of the form (1). Without loss of generality, we can write these systems in the following form
$$\begin{aligned} \dot{x} &= b_0(x) + b_1(x)y + \cdots + b_\ell(x)y^\ell, \\ \dot{y} &= a_0(x) + a_1(x)y + \cdots + a_n(x)y^n, \end{aligned} \tag{7}$$
where $a_i(x), b_j(x) \in \mathbb{C}[x]$ for $i = 1, 2, \ldots, n$, $j = 1, 2, \ldots, \ell$, $a_n(x)b_\ell(x) \neq 0$.

Our results characterize when a Liouvillian integrable system (7) possesses a finite invariant algebraic curve.

Theorem 1. *Consider a planar complex polynomial differential system (7) with a Liouvillian first integral. If $n > \ell + 1$ then the system has a finite invariant algebraic curve.*

The next proposition provides examples that when $n \leq \ell + 1$, the thesis of Theorem 1 does not hold.

Proposition 2. *Given $k \geq 0$ and $s > 0$ integers and k even, consider the polynomial differential systems*
$$\dot{x} = 2xy + y^k, \qquad \dot{y} = 1. \tag{8}$$
$$\dot{x} = y^k + (x+y)^{s-1}\big(-s + 2xy + 2y^2\big), \; \dot{y} = s(x+y)^{s-1}. \tag{9}$$
$$\dot{x} = s(x+y)^{s-1}, \; \dot{y} = x^k + (x+y)^{s-1}\big(-s + 2x^2 + 2xy\big). \tag{10}$$
$$\begin{aligned} \dot{x} &= s(x+y)^{s-1}\big(1 + 2x(x+y)^s\big), \\ \dot{y} &= 1 - s(x+y)^{s-1}\big(1 + 2x(x+y)^s\big). \end{aligned} \tag{11}$$
$$\begin{aligned} \dot{x} &= 1 - s(x+y)^{s-1}\big(1 + 2y(x+y)^s\big), \\ \dot{y} &= s(x+y)^{s-1}\big(1 + 2y(x+y)^s\big). \end{aligned} \tag{12}$$
These systems are Liouvillian integrable and have no finite invariant algebraic curves.

Proofs

Proof of Theorem 1.
This proof is by contradiction. We assume that the differential system (7) is Liouvillian integrable, i.e. has a Liouvillian first integral, and has no finite invariant algebraic curves. By the above cited result of Christopher [1] we know that if system (7) is Liouvillian integrable, then it has an integrating factor of the form (4). Therefore if system (7) is a planar Liouvillian integrable polynomial differential system without finite invariant algebraic curves, then it must have an integrating factor of the form $R = \exp(g(x,y))$, where g is a polynomial. We recall here that $g = 0$ does not need to be an invariant algebraic curve of system (7). We assume that the degree of g with respect to the variable y is m. Then we write g as a polynomial in the variable y with coefficients polynomials in the variable x, i.e. $R = \exp(g(x,y)) = \exp(g_0(x) + g_1(x)y + \cdots + g_m(x)y^m)$.

Now we impose that R is an integrating factor of system, i.e.,
$$\frac{\partial R}{\partial x}\dot{x} + \frac{\partial R}{\partial y}\dot{y} + \left(\frac{\partial \dot{x}}{\partial x} + \frac{\partial \dot{y}}{\partial y}\right)R = 0, \tag{13}$$
and we obtain the following identity
$$\begin{aligned} &\Big(g_0' + g_1'y + \cdots + g_m'y^m\Big)\Big(b_0 + b_1y + \cdots + b_\ell y^\ell\Big) + \\ &\Big(g_1 + 2g_2y + \cdots + mg_my^{m-1}\Big)\Big(a_0 + a_1y + \cdots + a_ny^n\Big) + \\ &{b_0}' + {b_1}'y + \cdots + {b_\ell}'y^\ell + a_1 + 2a_2y + \cdots + na_ny^{n-1} = 0, \end{aligned}$$
after dividing R. We have avoided the dependence on x of the functions a_i, b_j and g_k to simplify notation.
We are assuming that $n > \ell + 1$. So the highest power in y in the previous identity is y^{m+n-1} whose coefficient is ma_ng_m. Since a_n is, by definition, the coefficient of the highest power in y in $\dot{y}$, it cannot be zero and the same reasoning holds for g_m. Hence, $ma_ng_m = 0$ implies $m = 0$. Consequently the integrating factor is of the form $R = \exp(g_0(x))$. Therefore, from equation (13) we have that
$$g_0'\Big(b_0 + \cdots + b_\ell y^\ell\Big) + b_0' + \cdots + b_\ell' y^\ell + a_1 + \cdots + na_ny^{n-1} = 0.$$
Since $n > \ell + 1$ the highest power in y in the previous expression is y^{n-1} with coefficient na_n. The vanish of this coefficient leads to a contradiction because $n > 1$ by assumption and a_n cannot be null.

Proof of Proposition 2.
We first consider system (8), which has the inverse integrating factor $V(x,y) = e^{y^2}$. By the above cited result of Christopher [1], we have that it is Liouvillian integrable. We shall prove that it has no finite invariant algebraic curve. Assume that $f(x,y) = 0$ is a finite invariant algebraic curve and we write it expanded in powers of x: $f(x,y) = f_0(y) + f_1(y)x + \ldots + f_{n-1}(y)x^{n-1} + f_n(y)x^n$, where $f_n(y)$ is not identically zero and $f_i(y)$ are polynomials in $\mathbb{C}[y]$. We write its cofactor also expanded in powers of x: $k(x,y) = a_0(y) + a_1(y)x + \ldots + a_{m-1}(y)x^{m-1} + a_m(y)x^m$, where $a_i(y)$ are polynomials in $\mathbb{C}[y]$. Equation (2) writes as
$$\begin{aligned} &\Big(f_1 + 2f_2x + \ldots + (n-1)f_{n-1}x^{n-2} + nf_nx^{n-1}\Big)\left(2xy + y^k\right) \\ &+ \Big(f_0' + f_1'x + \ldots + f_{n-1}'x^{n-1} + f_n'x^n\Big) = (a_0 + a_1x + \ldots \\ &+ a_{m-1}x^{m-1} + a_mx^m\Big)\Big(f_0 + f_1x + \ldots + f_{n-1}x^{n-1} + f_nx^n\Big), \end{aligned}$$
where we have avoided the dependence on y to simplify notation. We observe that the highest order of x in the left–hand side is n and the highest order of x in the right–hand is $n+m$, which implies $m = 0$. Now we equate the highest powers in x of both sides, which correspond to the coefficient of x^n: $nf_n2y + f_n' = a_0\,f_n$. Since f_n is a polynomial in y, we deduce that $a_0 = 2ny$ and f_n needs to be a constant which we take equal to 1 without loss of generality. The equation corresponding to the coefficients of x^{n-1} is: $(n-1)f_{n-1}2y + ny^k + f_{n-1}' = 2ny\,f_{n-1}$. The integration of this linear differential equation gives $f_{n-1}(y) = ce^{y^2} + (n/2)e^{y^2}\Gamma\left((k+1)/2, y^2\right)$, where Γ denotes the Euler–Gamma function. Since k is even and f_{n-1} is a polynomial in y, we deduce $c = n = 0$, which means that the invariant algebraic curve can only depend on y. But since $\dot{y} = 1$, we get a contradiction.
The other four systems (9), (10), (11) and (12) are obtained by rational transformations of variables from system (8) and a rescaling of time, if necessary. Therefore, they all have a Liouvillian first integral which is the transformation of the first integral of system (8). In order to explicit the transformations without confusing the names of the variables, we write system (8) as $\dot{u} = 2uv + v^k$, $\dot{v} = 1$. The transformation $(u,v) \to (x,y)$ with $u = (x+y)^s$ and $v = y$ gives system (9). The other rational transformations can be found in [5]. We remark that if one of the systems (9), (10), (11) or (12) has a finite invariant algebraic curve $f(x,y) = 0$ by the rational change of variables, then equation $\frac{du}{dv} = 2uv + v^k$ has a particular algebraic solution. We recall here Theorem 3.1 of [2].

Theorem 3. *Consider system (1) and the corresponding ordinary differential equation $\frac{dy}{dx} = \frac{Q(x,y)}{P(x,y)}$. Let $\varphi(x)$ be an algebraic particular solution and we call $f(x,y)$ the irreducible polynomial satisfying $f(x,\varphi(x)) \equiv 0$. Then, the curve $f(x,y) = 0$ is an invariant algebraic curve of system (1).*

Since equation $\frac{du}{dv} = 2uv + v^k$ has no finite invariant algebraic curve, we conclude that systems (9), (10), (11) and (12) have no invariant algebraic curve.

References

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