On the approximation of periodic solutions of non-autonomous ordinary differential equations

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Introduction

In this poster we recover the pioneering results of Stokes [5] and Urabe [6] that provide a theoretical basis for proving that near truncated Fourier series, that approach a periodic solutions of a ordinary differential equation, there are actual periodic solutions of the equation. This result can be applied independently of the method that has been used to get such approximation. We present a couple of concrete examples coming from planar autonomous systems. In one of them we use the *Harmonic Balance Method* (HBM) to get an approximated solution while in the other we use a numerical approach.

Consider the real non-autonomous differential equation

$$x' = X(x,t), \tag{1}$$

where $X : \Omega\times[0,2\pi] \to \mathbb{R}$ is a $\mathcal{C}^2$-function, 2π-periodic in t, $\Omega\subset\mathbb{R}$ is a given open interval and the prime denotes the derivative with respect to t. Recall that any smooth 2π-periodic function $x(t)$ can be written as its Fourier series,

$$x(t) = \frac{a_0}{2} + \sum_{m=1}^{\infty}\left(a_m\cos(mt) + b_m\sin(mt)\right),$$

where

$$a_m = \frac{1}{\pi}\int_0^{2\pi} x(t)\cos(mt)\,dt, \quad \text{and} \quad b_m = \frac{1}{\pi}\int_0^{2\pi} x(t)\sin(mt)\,dt,$$

for all $m\geq 0$. Hence, it is natural trying to approximate the 2π-periodic solutions of the functional equation

$$\mathcal{F}(x(t)) := x'(t) - X(x(t),t) = 0, \tag{2}$$

by using truncated Fourier series, *i.e.* trigonometric polynomials. There are several ways of obtaining this truncated Fourier series, for instance, the HBM or simply by numerical approximation.

What is the HBM?

The HBM is an heuristic method used for finding periodic solutions of (1), or equivalently, periodic functions which satisfy the functional equation (2). The HBM of order N is defined as follows.
Consider a trigonometric polynomial

$$y_{\scriptscriptstyle N}(t) = \frac{r_0}{2} + \sum_{m=1}^{N}\left(r_m\cos(mt) + s_m\sin(mt)\right),$$

with unknowns $r_m = r_m(N), s_m = s_m(N)$ for all $m \leq N$. Then compute the 2π-periodic function $\mathcal{F}(y_{\scriptscriptstyle N}(t))$. It has also an associated Fourier series

$$\mathcal{F}(y_{\scriptscriptstyle N}(t)) = \frac{\mathcal{A}_0}{2} + \sum_{m=1}^{\infty}\left(\mathcal{A}_m\cos(mt) + \mathcal{B}_m\sin(mt)\right),$$

where $\mathcal{A}_m = \mathcal{A}_m(\mathbf{r},\mathbf{s})$ and $\mathcal{B}_m = \mathcal{B}_m(\mathbf{r},\mathbf{s})$, $m\geq 0$,
with $\mathbf{r} = (r_0, r_1, \ldots, r_{\scriptscriptstyle N})$ and $\mathbf{s} = (s_1,\ldots,s_{\scriptscriptstyle N})$.
The HBM consists in finding values $\mathbf{r}$ and $\mathbf{s}$ such that

$$\mathcal{A}_m(\mathbf{r},\mathbf{s}) = 0 \quad \text{and} \quad \mathcal{B}_m(\mathbf{r},\mathbf{s}) = 0 \quad \text{for} \quad 0\leq m\leq N. \tag{3}$$

The hope of the method is that the trigonometric polynomials found using this approach are "near" actual periodic solutions of the differential equation (1).

A simple example

Consider the planar ordinary differential equation

$$\begin{aligned} \dot x &= -y + x(a + dx^2 + exy + fy^2)\\ \dot y &= x + y(a + dx^2 + exy + fy^2) \end{aligned} \tag{4}$$

with $a=-1$, $d=3$ and $e=2$. In polar coordinates it writes as

$$\dot r = -r + (\cos(2t) + \sin(2t) + 2)r^3. \tag{5}$$

where we have renamed θ as t. The above equation is a Bernoulli equation that can be solved explicitly. Its solutions are $r(t)\equiv 0$ and

$$r(t) = \pm\frac{1}{\sqrt{2+\cos(2t) + ke^{2t}}}.$$

Therefore its unique periodic solution, which corresponds to a limit cycle of (4) is

$$r^*(t) = \frac{1}{\sqrt{2+\cos(2t)}}. \tag{6}$$

It is easy to prove that it is hyperbolic and unstable, see [3].

Let us forget that we know the exact solution to illustrate how to use the HBM for (5) to obtain an approximation to the periodic solution (6).

Following the HBM, we consider the functional equation

$$\mathcal{F}(r(t)) = r'(t) + r(t) - (\cos(2t) + \sin(2t) + 2)r^3(t) = 0. \tag{7}$$

We look for an approximation of the form $r(t) = r_0 + r_2\cos(2t)$. The vanishing of the coefficients of 1 and $\cos(2t)$ in the Fourier series of $\mathcal{F}(r(t))$ gives the nonlinear algebraic system:

$$r_0 - 2r_0^3 - \frac{3}{2}r_2r_0^2 - 3r_2^2r_0 - \frac{3}{8}r_2^3 = 0, \qquad r_2 - r_0^3 - 6r_2r_0^2 - \frac{9}{4}r_2^2r_0 - \frac{3}{2}r_2^3 = 0.$$

One of its approximate solutions is $\tilde r_0 \approx 0.7440$, $\tilde r_2\approx -0.20139$. Obviously, if we apply a higher order of the HBM, then the approach of the periodic solution is better. However sometimes it is not possible to solve the nonlinear algebraic system (3). In this example, we apply the HBM up to eighth-order. Thus, we obtain the candidate to be solution

$$\tilde r(t) = \sum_{k=0}^{4} r_{2k}\cos(2kt), \tag{8}$$

with $r_0 = 0.74574891$, $r_2 = -0.20168366$, $r_4 = 0.04065712$, $r_6 = -0.00909259$, and $r_8 = 0.00213382$

Is it possible to guarantee that near the calculated approximate solution by using the HBM, there is a periodic solution of the differential equation?

The affirmative answer to this question is given by Stokes and Urabe, see [5,6]. For knowing the result, we need introduce some definitions.

Definitions

Let $\bar x(t)$ be a 2π-periodic $\mathcal{C}^1$-function. We will say that

- $\bar x(t)$ is *noncritical* with respect to (1) if

$$\int_0^{2\pi}\frac{\partial}{\partial x}X(\bar x(t),t)\,dt \neq 0. \tag{9}$$

  Notice that if $\bar x(t)$ is a periodic solution of (1), then to be noncritical is equivalent to be *hyperbolic*.
- the *accuracy* of $\bar x(t)$ is

$$S := ||s(t)||_2 = \sqrt{\frac{1}{2\pi}\int_0^{2\pi}s^2(t)dt},$$

  where $s(t) := \mathcal{F}(\bar x(t)) = \bar x'(t) - X(\bar x(t),t)$.
  Notice that periodic solutions of (1) have accuracy 0.
- $M\in\mathbb{R}$ is a *deformation constant* associated to $\bar x(t)$ and X if

$$||y_b(t)||_\infty \leq M||b(t)||_2, \tag{10}$$

  where $y_b(t)$ is the unique 2π-periodic solution of the linear periodic system

$$y' = \frac{\partial}{\partial x}X(\bar x(t),t)\,y + b(t),$$

  $b(t)$ is a smooth 2π-periodic function, and $||f||_\infty = \max_{x\in\mathbb{R}}|f(x)|$.

Finally, we need to give a bound for the second derivative. Given

$$I := [\min_{\{t\in\mathbb{R}\}}\bar x(t) - 2MS, \max_{\{t\in\mathbb{R}\}}\bar x(t) + 2MS] \subset\Omega,$$

let $K<\infty$ be a constant such that

$$\max_{(x,t)\in I\times[0,2\pi]}\left|\frac{\partial^2}{\partial x^2}X(x,t)\right| \leq K. \tag{11}$$

Next theorem improves the result of Stokes and Urabe in the one-dimensional setting. More concretely, they prove the existence and uniqueness of the periodic orbit when $4M^2KS<1$. We give a similar proof with the small improvement $2M^2KS<1$; see [1]. Moreover our result gives, under an additional condition, the hyperbolicity of the periodic orbit.

Theorem (Stokes, Urabe):

Let $\bar x(t)$ be a 2π-periodic, $\mathcal{C}^1$–function such that

- it is noncritical with respect to (1),
- its accuracy with respect to (1) is S
- M is the deformation constant associated to $\bar x(t)$ and X
- given I there exist K satisfying (11).

Therefore, if $2M^2KS<1$, there exists a 2π-periodic solution $x^*(t)$ of (1) satisfying $||x^*-\bar x||_\infty\leq 2MS$, and it is the unique periodic solution of the equation entirely contained in this strip.

In addition, if the integral (9) in absolute value is bigger than $2\pi/M$ then, the periodic orbit $x^*(t)$ is hyperbolic, and its stability is given by the sign of the integral.

Return to the example

Once we have the approximate solution (8), we need to verify that the theorem applies.
Computing the accuracy of $\tilde r(t)$ we obtain that it is 0.0039. Because of the rational number of the coefficients make more difficult the subsequent computations, we approximate all the coefficients of (8) by suitable convergents of their respective expansions in continuous fractions. This is done in such a way that using these new coefficients we obtain a new approximate solution with similar accuracy.
For instance some convergents of r_0 are $1, 2/3, 3/4, 41/55, 44/59,\ldots$ and we choose $44/59$ and we do this for each r_{2k}. Thus, we can consider as an approximation of the periodic solution

$$\bar r(t) = \frac{44}{59} - \frac{24}{119}\cos(2t) + \frac{2}{49}\cos(4t) - \frac{1}{110}\cos(6t) + \frac{1}{468}\cos(8t). \tag{12}$$

The accuracy of $\bar r$ is $S = ||\mathcal{F}(\bar r(t))||_2 = 0.0039$. By calculating the deformation constant M and K we obtain $M=2.4$ and $K=20.91$. Finally, $2M^2KS\approx 0.96<1$ and the Theorem can be applied.

Therefore, we have proved the following

Proposition. Consider the periodic function $\bar r(t)$ given in (12). Then there is a periodic solution $r^*(t)$ of (5), such that

$$||\bar r - r^*||_\infty \leq 0.0192,$$

which is hyperbolic and unstable and it is the only periodic solution of (5) in this strip.

Remark: Using the analytic expression of $r^*(t)$ given in (6) it can be seen that indeed

$$||\bar r - r^*||_\infty\leq 0.0007.$$

A rigid cubic system

The second example corresponds to the rigid cubic system

$$\begin{aligned}\dot x &= -y + \tfrac{x}{10}(1 - x - 10\,x^2),\\ \dot y &= x + \tfrac{y}{10}(1 - x - 10\,x^2),\end{aligned}$$

that in polar coordinates writes as

$$\dot r = \frac{1}{10}r - \frac{1}{10}\cos(t)\,r^2 - \cos^2(t)\,r^3, \tag{13}$$

which has a unique positive periodic orbit, see [2].
In this example, we found computational difficulties to obtain the third-order approximation given by the HBM. So, we got numerically the approximation to the periodic solution; then, we computed, also numerically, the first terms of its Fourier series, and finally, we used again the continued fraction expansions to simplify the values appearing in our computations. We have proved:

Proposition. Consider the periodic function

$$\bar r(t) = \frac{4}{9} - \frac{1}{693}\cos(t) - \frac{1}{51}\sin(t) - \frac{1}{653}\cos(2t) - \frac{1}{45}\sin(2t) - \frac{1}{780}\cos(3t).$$

Then, the differential equation (13) has a periodic solution $r^*(t)$, such that

$$||\bar r - r^*||_\infty\leq 0.042,$$

which is hyperbolic and stable and it is the only periodic solution of (13) contained in this strip.

References

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