Symplectic Surface Diffeomorphisms

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Joint Work with Michael Handel

Suppose M is a compact connected oriented surface.

Definition

$\mathrm{Diff}^r(M)$ *denotes the* C^r *diffeomorphisms isotopic to the identity; if* $r = \omega$ *this denotes real analytic diffeos.*
$\mathrm{Symp}^r_\mu(M)$ *denotes the symplectic diffeos, the subgroup of* $\mathrm{Diff}^r(M)$ *which preserve the volume form* μ.

Normal Solvable Subgroups

Theorem

Suppose M is a compact oriented surface of genus 0 and G is a subgroup of $\mathrm{Symp}^{\omega}_{\mu}(M)$. Suppose further that G has an infinite normal solvable subgroup. Then G is virtually abelian.

Corollary

Suppose M is a compact surface of genus 0 and G is a solvable subgroup of $\mathrm{Symp}^{\omega}_{\mu}(M)$, then G is virtually abelian.

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Centralizers

Definition

We will denote by $Cent^r(f)$, *the centralizer of* f, *the subgroup of* $\mathrm{Diff}^r(M)$ *whose elements commute with* f, *and by* $Cent^r_\mu(f)$ *the subgroup of* $\mathrm{Symp}^r_\mu(M)$ *whose elements commute with* f.

Corollary (F - Handel)

Suppose $f \in \mathrm{Symp}^\omega_\mu(S^2)$ *has infinite order, then* $Cent^\omega_\mu(f)$, *the centralizer of* f *in* $\mathrm{Symp}^\omega_\mu(S^2)$ *is virtually abelian.*

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The Centralizer of $f \in \mathrm{Diff}(M)$

Theorem (Bonatti, Crovisier, Wilkinson)

The C^1 generic $f \in \mathrm{Diff}^(M)$ has infinite cyclic centralizer.

Theorem (Farb-Shalen)

Suppose $f \in \mathrm{Diff}^\omega(S^1)$ has infinite order, then $Cent^\omega(f)$, the centralizer of f in $\mathrm{Diff}^\omega(M)$, is virtually abelian.

Question

Suppose M is a closed surface and $f \in \mathrm{Diff}^\omega(M)$ has infinite order. Then is its centralizer, $Cent^\omega(f)$, always virtually abelian?

Our second Corollary answers this in the case $f \in \mathrm{Symp}^\omega_\mu(M)$

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Three types of structure for $f \in \mathrm{Symp}^\infty_\mu(M)$

Let M be a compact oriented surface with genus zero and let G be a subgroup of $\mathrm{Symp}^\infty_\mu(M)$.

- G contains an element of positive entropy
- G contains an element f which is multi-rotational, i.e. if $M = S^2$, then f has entropy 0 and at least three periodic points.
- G is a pseudo-rotation group.

These exhaust the possibilities.

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Theorem (Katok)

Suppose $f \in \mathrm{Diff}^{1+\epsilon}(M^2)$ *has positive topological entropy. Then there is a hyperbolic periodic point* p *for* f *with a transversal homoclinic point.*

Corollary (Katok)

Suppose $f \in \mathrm{Diff}^2(M^2)$ *has positive topological entropy, then* $Cent^{\omega}(f)$*, the centralizer of* f *in* $\mathrm{Diff}^{\omega}(M)$*, is virtually cyclic. Moreover, every infinite order element of* $Cent^{\omega}(f)$ *has positive topological entropy.*

Positive Entropy

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Proof of Corollary

Lemma

Suppose $f \in \mathrm{Diff}^2(M^2)$, $g \in Cent^2(f)$, and f has a hyperbolic fixed point p of saddle type. If g fixes p and preserves the branches of $W^s(p, f)$, then there is a C^1 coordinate function t on $W^s(p, f)$ and a unique number $\alpha > 0$ such that in these coordinates $g(t) = \alpha t$. In particular α is an eigenvalue of Dg_p.

Proof.

Sternberg linearization says there is a C^1 coordinate function t on $W^s(p, f)$ in which $f(t) = \lambda t$.

Then if $g \in Cent^2(f)$,

$$g(t) = \lambda^{-n} g(\lambda^n t),$$

so

$$g'(t) = g'(\lambda^n t) = g'(0).$$

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$$\phi : Cent^r(f,p) \to \mathbb{R}^+,$$

is defined by $\phi(g) = \alpha$ *where* α *is the number for which* $g(x) = \alpha x$ *It is a homomorphism.*

To prove the corollary we need only show that ϕ is injective and it has discrete image.

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- If $\phi(h) = 1$, then $W^s(f, p) \subset \mathrm{Fix}(h)$.
- The set of fixed points of an analytic diffeomorphism $h : M^2 \to M^2$ is an analytic set which implies it has finitely many components and its complement has finitely many components (this is true even in a chart).
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Definition

Suppose M is a compact genus zero surface and $f \in \mathrm{Symp}^\infty_\mu(M)$ and that the number of periodic points of f is greater than the Euler characteristic of M. If f has infinite order and entropy 0, we will call it a multi-rotational diffeomorphism. *This set of diffeomorphisms will be denoted $\mathcal{Z}(M)$.*

Definition

Annular compactification of an annulus U: There is a dynamically compatible compactification of any f-invariant annulus. It is the blowup on an end whose frontier is a single point and the prime end compactification otherwise.

Multi-Rotational Diffeomorphisms

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The Structure of Multi-Rotational Diffeomorphisms.

Theorem (F - Handel)

Suppose $f \in \mathrm{Symp}^\infty_\mu(S^2)$ has infinite order, entropy 0, and at least three periodic points (i.e., f is multi-rotational). Let $\mathcal{A} = \mathcal{A}_f$ be the collection of maximal f-invariant open annuli in $S^2 \setminus \mathrm{Fix}(f)$, then

1. *The elements of $\mathcal{A}$ are pairwise disjoint.*
2. *The union $\bigcup_{U \in \mathcal{A}} U$ is a full measure dense open subset of $S^2 \setminus \mathrm{Fix}(f)$.*
3. *Each component of the frontier of U in S^2 contains a fixed point.*
4. *The rotation number $\rho_f : U_c \to S^1$ is continuous and non-constant. Each component of the level set of ρ_f which is disjoint from ∂U_c is essential in U, i.e. separates the components of ∂U_c.*

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Three Key Properties

- The elements of $\mathcal{A}$ are permuted by any $g \in Cent^\infty_\mu(f)$.
- The $Cent^\infty_\mu(f)$-orbit of any $U \in \mathcal{A}$ is finite.
- If $g \in Cent^\infty_\mu(f)$ preserves U then it preserves all components of level sets of ρ_f.

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Mean rotation number

If $f : A \to A$ is a closed annulus let $\Delta_f(x) = p_1(\tilde{f}(\tilde{x})) - p_1(\tilde{x})$. Then the mean rotation number $\rho_\mu : \mathrm{Homeo}_\mu(A) \to S^1$ is the coset mod $\mathbb{Z}$ of

$$\int_A \Delta_f(x)\, d\mu.$$

It is the average rotation number or the "flux" across a line joining the two boundary components of A. It is a homomorphism and hence if $h = [g_1, g_2]$ for some $g_i : A \to A$ then $\rho_\mu(h) = 0$.

Theorem

If $\rho_\mu(f) = 0$ then f has a fixed point in the interior of A.

Theorem (F - Handel)

Suppose f is multi-rotational. Then $Cent^{\omega}_{\mu}(f)$*, the centralizer of f in* $\mathrm{Symp}^{\omega}_{\mu}(S^2)$ *is virtually abelian.*

Proof:

1) By the structure theorem for multi-rotational f there is $U \in \mathcal{A}(f)$. Let $Cent(U)$ be the (finite index) stabilizer of U in $Cent^{\omega}_{\mu}(f)$.
2) Let $g_1, g_2 \in Cent(U)$ and let $h = [g_1, g_2]$. We will contradict $h \neq id$ by showing $\mathrm{Fix}(h)$ has infinitely many components. This will show $Cent(U)$ is abelian.

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Proof Cont'd

3) Choose components of the level sets of ρ_f, $C_1, C_2, \ldots$ satisfying
   - For each i, $\rho_f(C_i)$ is irrational.
   - For each i, C_i separates C_{i+1} from $\bigcup_{j<i} C_j$.
4) Let A_i denote the open subannulus of U whose frontier is $C_i \cup C_{i+1}$. Then A_i is $Cent(U)$-invariant and hence h has a fixed point in A_i (since h is the commutator of $g_1, g_2 : A_i \to A_i$).
5) Choose a $V \in \mathcal{A}(h)$ which intersects A_i and let W be a component of $V \cap A_i$. There are three subcases each of which leads to a contradiction:
   - (a) W is a disk;
   - (b) W is essential in A_i;
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Pseudo-Rotation Groups

Definition

Suppose M is a compact oriented surface with Euler characteristic $\mathfrak{X}(M) \geq 0$, i.e. M is $S^2, \mathbb{A}$ or D^2. A pseudo-rotation subgroup *of $\mathrm{Symp}^r_\mu(M)$ with $r \geq 1$, is a subgroup G with the property that every non-trivial element of G has exactly $\mathfrak{X}(M)$ fixed points.*

One can show that if $M = \mathbb{A}$ or D^2 then any pseudo-rotation subgroup of $\mathrm{Symp}^r_\mu(M)$ is abelian.

Question

Must a pseudo-rotation subgroup of $\mathrm{Symp}^\omega_\mu(S^2)$ be conjugate to a subgroup of $SO(3)$? Must a pseudo-rotation subgroup of $\mathrm{Symp}^\omega_\mu(S^2)$ be isomorphic to a subgroup of $SO(3)$?

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Tits Alternative

Recall that the Tits alternative is satisfied by a group G if every subgroup (or perhaps every finitely generated subgroup) of G is either virtually solvable or contains a non-abelian free group. This is a deep property known for finitely generated linear groups and some groups arising in geometric group theory. It is an important open question for $\mathrm{Diff}^{\omega}(S^1)$. (It is not true for $\mathrm{Diff}^{\infty}(S^1)$.)

Tits Alternative

Conjecture (Tits alternative)

If M is a compact surface then every finitely generated subgroup of $\mathrm{Symp}^{\omega}_{\mu}(M)$ is either virtually solvable or contains a non-abelian free group.

Theorem

Suppose M is a compact genus zero surface and G is a subgroup of $\mathrm{Symp}^{\omega}_{\mu}(M)$. If G contains at least one multi-rotational element then either G contains a subgroup isomorphic to F_2, the free group on two generators, or G has an abelian subgroup of finite index.

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THANK YOU!