New Trends in Dynamical Systems
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Algebraic Moments : from Abel equations to Jacobian conjecture

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I-Analytic extensions of perturbative series

I-1-The example of moment series and 1-dim non autonomous differential equations of Abel type.

Algebraic (trigonometric) moments appeared in the context of a perturbative analysis of Abel equations :

$$\frac{dy}{dx} = p(x)y^3 + q(x)y^2, \qquad (1)$$

We say that the Abel equation displays a center between point a and b if the solution $y(x, y_0)$ satisfies $y(a, y_0) = y(b, y_0)$ for all y_0. The problem was to characterize the centers.

We considered from the begining the parametric (or persistent center) problem. Find the Abel equation so that for all ϵ small,

$$\frac{dy}{dx} = p(x)y^3 + \epsilon q(x)y^2, \tag{2}$$

displays a center. We derived easily a necessary condition :

$$\int_a^b P(x)^j dQ(x) = 0 \tag{3}$$

Polynomial moment problem : For $P(z) \in \mathbb{C}[z]$ and $a, b \in \mathbb{C}, a \neq b$ describe $Q(z) \in \mathbb{C}[z]$ such that :

$$\int_a^b P^i(z) dQ(z) = 0,$$

for all $i \geq 0$.

Example 1. $P(z) = z, [a, b]$.
There exists no non-constant solutions by the Weierstrass theorem.

Example 2. $P(z) = z^2, [-1, +1]$.
There exist non-constant solutions e.g. $Q(z) = R(z^2), R(z) \in \mathbb{C}[z]$.

Example 3. $P(z)$ satisfies $P(a) = P(b))$.
There exist non-zero solutions e.g.

$$Q(z) = R(P(z)), R(z) \in \mathbb{C}[z].$$

Proof :

$$\int_a^b P^i(z)dQ(z) = \int_{P(a)}^{P(b)} y^i dR(y) = 0$$

since $P(a) = P(b)$ and $R(y)$ is holomorphic in $\mathbb{C}$.

If there exist $\tilde{P}(z), W(z) \in \mathbb{C}[z]$ such that

$$P(z) = \tilde{P}(W(z)), W(a) = W(b)$$

then for any $\tilde{Q}(z) \in \mathbb{C}[z]$ the polynomial

$$Q(z) = \tilde{Q}(W(z)),$$

is a solution. Such a solution is called composed.

Proof :

$$\int_a^b P^i(z)dQ(z) = \int_{W(a)}^{W(b)} \tilde{P}^i(y)d\tilde{Q}(y) = 0.$$

Question : Is it true that any solution is composed ?

Theorem. (C. Christopher, 2000) *If* $P('a) \neq 0, P'(b) \neq 0$ *then any non-constant solution is composed.*

Pakovich, 2001 Let $T_n(z)$ be the n^{th} Chebyshev polynomial, $T_n(\cos z) = \cos(nz)$, then

$$T_n(T_m(z)) = T_{mn}(z) = T_m(T_n(z)).$$

Take

$$P(z) = T_6(z), a = -\sqrt{3}/2, b = \sqrt{3}/2, Q(z) = T_3(z) + T_2(z).$$

C. Christopher Abel equations : composition conjectures and the model problem. Bull. London Math. Soc. 32 (2000), no. 3, 332-338.

Assume

$$\int_a^b P^i(z)dQ(z) = 0,$$

for all $i \geq 0$ and $P'(a) \neq 0, P'(b) \neq 0$.

Without loss of generality, take P monic, $a = 0, b = 1, P(0) = 0$. If

$$|c| > K := \sup_{x\in[0,1]}|P(x)|,$$

$(P(x) - c)^{-1}$ has a well-defined expansion :

$$I(c) = \int_0^1 \frac{q(x)}{P(x) - c}dx = \Sigma_{i=0}\frac{1}{c^{i+1}}\int_0^1 P(x)^i q(x)dx.$$

Hypothesis is equivalent to $I(c) = 0$.

Let $S = \{c_0, c_1, ..., c_n\}$ the critical values of P to which we add $c_0 = P(0) = 0, c_1 = P(1)$. As $c \in \mathbb{C} - S$, $P(x) - c$ has distinct roots, none of which is 0 or 1.

Let $\alpha_i(c)$ be the roots of $P(x) - c = 0$,

$$\frac{q(x)}{P(x) - c} = r(x, c) + \sum_i \frac{m(\alpha_i(c))}{x - \alpha_i(c)},$$

where r is a polynomial in x and c and $m(x) = \frac{q(x)}{p(x)}$.

$$I(c) = R(c) + \sum_i m(\alpha_i(c)) \ln_i(1 - \frac{1}{\alpha_i}).$$

As seen above, $I(c)$ is multi-valued in general. The main idea is that if $I(c) \equiv 0$ then its “periods” (differences between two branches) are zero. With little further classical tools (Lüroth theorem), this allows to conclude in this case.

General Case

In 2009 Pakovich and Muzychuk proved the general final result :

Theorem

Two polynomials $P(x), Q(x) \in \mathbb{R}[x]$ and $a, b \in \mathbb{R}, a \neq b$, satisfies :

$$\int_a^b P^i(x) dQ(x) = 0,$$

for all $i \geq 0$ if and only if there are finitely many polynomials $W_1, W_2, ..., W_s$, $\tilde{P}_1, ...\tilde{P}_s$, $\tilde{Q}_1, ..., \tilde{Q}_s$ such that :

$$P(x) = \tilde{P}_i(W_i(x)), i = 1, ..., s$$

$$Q(x) = \Sigma_i \tilde{Q}_i(W_i(x)),$$
$$W_i(a) = W_i(b).$$

The method used in the proof consists in analyzing the rational combinaison of the values of $m = \frac{q}{p}$ on the different branches of the algebraic function Q^{-1}.

Return to the initial center focus problem and the trigonometric moment problems

J. Giné, M. Grau, J. Llibre "Universal centers and composition conditions" J. of the London Mathematical Society, advanced access 09/11/12,

Universal center (all iterated integrals vanish) is equivalent to composition

A. Cima, A. Gasull, F. Mañosas "A simple solution of some composition conjectures for Abel equations" J. Math. Ann. and Appl. (2012).

The approach is completely novel and yields for both polynomial and trigonometric to the equivalence between :

Composition

strong persistence

Double moments vanishing (with precise bound on the sufficient number of double moments to vanish).

Some open questions :

For polynomial Abel equations, no example is known yet of a center which is not persistent (and hence composed). Such an example was given in the trigonometric case by Cima, Gasull and Mañosas.

I-2 "Global" Birkhoff normal forms

G. D. Birkhoff studied the local expression of a Hamiltonian system near a critical point of Morse type up to symplectic changes of coordinates. Under some generic conditions, this local normal form exists as a formal series in any dimension. It is convergent in one degree of freedom and it is generically divergent in $(m > 1)$ degrees of freedom [(Siegel, 54)(Moser, 76)]

Hamiltonian systems integrable in Liouville sense

A Hamiltonian system (H,ω) is said to be (completely) integrable if there exist m generically independent integrals $\underline{H} = (H_1, ..., H_m)$ such that $\{H_i, H_j\} = 0$, $H = H_1$. If the fibers $\underline{H}^{-1}(c)$ are compact and connected, they are torii and the flows of all the H_i are linear on these torii. Action-angles coordinates allow to compute the frequencies of the Hamiltonian flow of H on these invariant torii.

Analytic Hamiltonian systems which are Liouville integrable display a convergent Birkhoff normal form

Theorem (J. Vey, 76)

Assume $\underline{H}$ are analytic near $0 \in \mathbb{R}^n$, $\{H_i, H_j\} = 0$, $H = H_1$ displays a Morse critical point, assume that the $\mathrm{Hess} H_i(0)$ generate a Cartan sub-algebra of $Sp(2m, \mathbb{R})$, then the Birkhoff normal form of H is a convergent series.

For instance, $p_i = x_i^2 + y_i^2$ or $p_i = x_i y_i$ generate a Cartan sub-algebra of $Sp(2m, \mathbb{R})$ (Precise definition : commutative and selfnormalizing).

Birkhoff normal form of an integrable Hamiltonian system is in general a convergent series but *a priori* only defined in the neighborhood of the critical point. What can be said of its analytic prolongation if the Hamiltonian system is itself globally defined (for instance is a polynomial, rational function) ?

In such case, if the HS displays different critical points, is it possible to compare the analytic prolongation of the Birkhoff form in one critical point to the Birkhoff form in the other critical point ?

Begin with one degree of freedom !

The pendulum has been studied recently (P.L.Garrido, G. Gallavotti, JPF, Journal Maths Physics 10).

Reading Jacobi in the text and Gradshtein-Ryzhik tables of formula for elliptic functions.

Jacobi found a coordinate system in which the motion is linear but he did not computed the symplectic form in these coordinates.

Although his computation could be used to obtain a piece of information on the symplectic form (its relative cohomology class associated with H).

Singularity theory of functions

Consider an analytic function $H : (x,y) \mapsto \frac{1}{2}(x^2+y^2)+ ....$ and a symplectic (volume) form $\omega = dx \wedge dy$. Morse lemma allows to find anew (analytic) coordinate system (X,Y) such that $H=\frac{1}{2}(X^2+Y^2)$ but with no control of ω : $\omega = [1+F(X,Y)]dX \wedge dY$.

Definition Two volume forms ω and ω' have the same relative cohomology class if $\omega - \omega' = dH \wedge d\xi$.

Moser isotopy method for volume form applies and shows : Given a function H and two volume forms which are relatively cohomologous, there is an isotopy ϕ so that $\phi * (H) = H$ and $\phi * (\omega) = \omega'$.

Any polynomial form ω decomposes into $\omega = \psi(H) dx \wedge dy + dH \wedge d\xi$.

After these changes of coordinates we find $H = \frac{1}{2}(x^2 + y^2), \omega = \psi(H) dx \wedge dy$, there is and easy change into,

$$H = \phi(\frac{1}{2}(x^2 + y^2)), \omega = dx \wedge dy$$

and this is the Birkhoff normal form !

Free rigid body motion and the geodesic motion on a revolution ellipsoid have been also studied (P.L. Garrido, G. Gallavotti, JPF, F18 in Ipparco Roma I, 2012.)

Normal forms are derived via the analysis of relative cohomology.

The Hamiltonian of the free rigid body in the coordinates B, β depending of the parameters I and of the "parameter" A(in fact constant of motion) :

$$H' = \frac{1}{2}\frac{B^2}{I_3} + \frac{1}{2}(\frac{\cos^2\beta}{I_1} + \frac{\sin^2\beta}{I_2})((A^2 - B^2), \tag{4}$$

We consider instead :

$$H = \frac{2H' - A^2 I_1^{-1}}{I_3^{-1} - I_1^{-1}} = B^2 - r^2 \sin^2\beta A^2 + r^2 \sin^2\beta B^2, \tag{5}$$

with

$$r^2 = \frac{I_1^{-1} - I_2^{-1}}{I_3^{-1} - I_2^{-1}}. \tag{6}$$

Consider the Hamiltonian :

$$H = B^2 + r^2\beta^2 A^2 + r^2(\sin^2\beta - \beta^2)A^2 - r^2\sin^2\beta B^2. \tag{7}$$

Change (B, β) into $(X = B, Y = rA\beta)$, this multiplies the symplectic form by $1/rA$ and this yields to consider in the following the couple :

$$H = X^2 + Y^2 + ..., \omega = dX \wedge dY \tag{8}$$

We have here an explicit version of the Morse lemma :

$$\begin{aligned} &(X,Y) \mapsto (X',Y') \\ &X' = X\sqrt{1 - r^2\sin^2(\tfrac{Y}{rA})} = X + ...h.o.t. \\ &Y' = rA\sin(\tfrac{Y}{rA}) = Y + ...h.o.t. \end{aligned} \tag{9}$$

In these coordinates :

$$\begin{aligned} &H = X'^2 + Y'2, \\ &\omega = dX \wedge dY = \frac{1}{\sqrt{(1-\frac{Y'2}{A^2})(1-\frac{Y'2}{r^2A^2})}} dX' \wedge dY'. \end{aligned} \tag{10}$$

We compute the cohomology class of the volume form ω by introducing the coefficients a_k defined as follows :

$$\frac{1}{\sqrt{1-u^2}} = \Sigma_k a_k u^{2k}, \tag{11}$$

and the binomial coefficients C_n^k. One can check that

$$\omega = g(\xi) dX' \wedge dY' + d\xi \wedge du, \tag{12}$$

where

$$g(\xi) = \Sigma_h (\Sigma_{k,l;k+l=h} a_k a_l r^{2h}) \frac{C_{2h}^h}{4^h} \frac{\xi^h}{r^{2h}}. \tag{13}$$

It remains to perform a final change of coordinates of type :

$$x = X"u(\xi), \quad y = Y"u(\xi),$$

so that $U(\xi) = u^2(\xi)$ satisfies :

$$U(\xi) + \xi U'(\xi) = g(\xi).$$

Note that this last equation is easily solved in terms of formal series, if $g(\xi) = \Sigma_n g_n \xi^n$, then $U(\xi) = \Sigma_n \frac{g_n}{n+1}$.Inverting the series : $\xi = HU(H)$:

$$U(H) = \Sigma_h (\Sigma_{k+l=h} a_k a_l r^{2l}) \frac{C_{2h}^h}{4^h(h+1)} \frac{H^h}{r^{2h}}$$

into $H = \xi V(\xi)$ yields the Birkhoff normal form.

The coefficients of the Birkhoff normal form are polynomials in a variable r depending on the inertia moments. We checked numerically that their roots are on the unit circle. We proved that this is true for the series issued from the cohomology class and discuss the link with D. Ruelle's articles on the extensions of the Lee-Yang theorem.

II-The Jacobian conjecture

II-1. Short history of the Jacobian conjecture and of the Markus-Yamabe conjecture

Arno van den Essen "Polynomial automorphisms and the Jacobian conjecture" vol 190, Pr. in Maths, Birkhauser (2000).

de Bondt-van den Essen "Nilpotent symmetric Jacobian Matrices and the Jacobian conjecture" J. Pure and Appl. Alg. 193 (2004) and 196 (2005). Independently by Meng (2006)

Wenhua Zhao "Hessian nilpotent polynomials and the jacobian conjecture" Transactions of the AMS, 359, (2007), 249-274 : Vanishing conjecture is equivalent to Jacobian conjecture.

Olivier Mathieu " Some conjectures about invariant theory and their applications", Algèbre non commutative, groupes quantiques et invariants (Reims, 1995), Sémin. Congr., vol. 2, Soc. Math. France, Paris, 1997, 263–279.

Wenhua Zhao "Generalizations of the image conjecture and the Mathieu conjecture", J. Pure Appl. Algebra 214 (2010), 1200–1216.
"Images of commuting differential operators of order one with constant leading coefficients", Journal of Algebra 324 (2010), 231–247.

Arno van den Essen, The amazing Image Conjecture, http ://arxiv.org/abs/1006.5801, 2010.

van den Essen, Arno ; Wright, David ; and Zhao, Wenhua, "On the Image Conjecture" (2012)

The Markus-Yamabe conjecture and a problem of Lasalle

MYC : Let $F : \mathbb{R}^n \to \mathbb{R}^n$ be a C^1 vector field with $F(0) = 0$ satisfying the MYA : for all $x \in \mathbb{R}^n$, the real parts of all eigenvalues of $JF(x)$ are negative, then 0 is a global attractor. WMYC : same hypothesis then F is injective.

Lasalle problem : Let $F : \mathbb{R}^n \to \mathbb{R}^n$ be a C^1 map satisfying the DMYA : for all x,all eigenvalues of $JF(x)$ are of absoulte value less than 1. Does $F(0) = 0$ implies that 0 is a global attractor of F ?

Fixed point conjecture : For all n, every polynomial map of $\mathbb{R}^n$ to itself satisfying DMYA has a unique fixed point.

A. Gasull, J. Llibre and J. Sotomayor "Global Asymptotic stability of Differential Equations in the plane, JDE, 1991.

Pinchuk "A counterexample to the strong real Jacobian" Math. Z. (An example of polynomial mapping with a non-vanishing Jacobian which is not a global diffeomorphism), 1994.

J. Bernat and J. Llibre "Counterexample to Kalman and Markus-Yamabe conjectures in dim larger than 4" Dyn. Cont. Disc. and Impulsive Systems 2(1996) 337-379.

Cima, van den Essen, Gasull, Hubbers and Mañosas "A polynomial counterexample to the Markus-Yamabe conjecture" Advances in Maths, 131 (1997), 453-457.

Cima, Gasull, Mañosas "The discrete Markus-Yamabe problem" Nonlinear Analysis (1999) : The fixed point conjecture is equivalent to the Jacobian conjecture.

II-2 Vanishing conjecture, Mathieu's conjectures and Image conjectures

After the reduction proposed by de Bondt and van den Essen (2004-2005), the Jacobian conjecture can be reduced to considering

$$F : z \mapsto w, w_i = z_i - \frac{\partial P}{\partial z_i}, \tag{14}$$

Zhao proposed to characterize these inverses as follows.

Theorem 1. *Let t be a parameter, consider the deformation $F_t(z) = z - t\nabla P$. The inverse map of $z \mapsto F_t(z)$ can be written :*

$$G_t(z) = z + t\nabla Q_t(z), \tag{15}$$

where $Q_t(z)$ is the unique solution of the Cauchy problem for the Hamilton-Jacobi equation :

$$\begin{array}{l} \frac{\partial Q_t(z)}{\partial t} = \frac{1}{2} < \nabla Q_t, \nabla Q_t >, \\ Q_{t=0}(z) = P(z) \end{array} \tag{16}$$

Note that with $U_t = \nabla Q_t$, the equation can we alternatively written :

$$\frac{\partial U_t(z)}{\partial t} = J(U_t(z))U_t(z), \tag{17}$$

which is the inviscid n-dimensional Burgers equation.

Proof. More generally, the formal inverse of $F_t(z) = z - tH(z)$ is the formal series $G_t(z) = z + tN_t(z)$ if and only if :

$$\begin{array}{l} N_t(F_t(z)) = H(z), \\ H(G_t(z)) = N_t(z). \end{array} \tag{18}$$

This yields the equations :

$$0 = \frac{\partial}{\partial t}[N_t(F_t(z))], \tag{19}$$

$$0 = \frac{\partial N_t}{\partial t}(F_t(z)) + J(N_t(z))\frac{\partial F_t}{\partial t}, \tag{20}$$

$$0 = \frac{\partial N_t}{\partial t}(F_t(z)) - J(N_t(z))H. \tag{21}$$

After composing with $G_t(z)$ from the right, this displays :

$$\frac{\partial N_t}{\partial t} = J(N_t)H(G_t) = J(N_t)N_t. \tag{22}$$

The theorem is proved in particular in the gradient case where $\nabla Q_t(F_t) = \nabla P$ and $\nabla P(G_t) = \nabla Q_t$ as a consequence of :

$$\frac{\partial}{\partial z_i}(\frac{\partial Q_t}{\partial t}) = \Sigma_j \frac{\partial^2 Q_t}{\partial z_i \partial z_j}\frac{\partial Q_t}{\partial z_j} = \frac{\partial}{\partial z_i} < \nabla Q_t, \nabla Q_t > . \tag{23}$$

□

We say that P is HN (Hessian nilpotent) if its Hessian matrix is Nilpotent. Recall the notation :

$$F_t(z) = z - t\nabla P, G_t(z) = z + t\nabla Q_t(z). \tag{24}$$

We can prove successively :

$$\Delta Q_t(F_t) = \Sigma_{k=1}^{\infty} t^{k-1} Tr[Hess^k(P)] \tag{25}$$

P is HN if and only if Q_t is Harmonic.

Let $a = \{a_I, I \in \mathbb{N}^n, \mid I \mid \geq 2$ be a set of commuting variables and let $P = \Sigma a_I z^I$ be the universal power series in z. For any k define the ideal U_k generated in $\mathbb{C}[a]$ by all coefficients of $\{u_m(P) = TrHess^m(P)\}, m = 1, ..., k$ and V_k the ideal generated by the coefficients of $\{v_m(P) = \delta^m P^m\}, m = 1, ..., k$. then for all k, $U_k = V_k$. So we deduce that P is HN iff and only if $\Delta^m P^m = 0$. Then $\mathsf{Q}_t(z) = \Sigma_{m=1}^{\infty} \frac{t^{m-1}}{2^{m-1} m!(m-1)!} \Delta^{m-1}(P^m)$ From these lemmas, follows the

Theorem 2. *The Jacobian conjecture is equivalent to the vanishing conjecture : for all polynomial P homogeneous of degree 4, if $\Delta^m(P^m) = 0$ for all m then $\Delta^{m-1} P^m = 0$ for all $m >> 0$.*

Olivier Mathieu proposed a general conjecture which is equivalent to the Jacobian conjecture :

Let K be a compact connected Lie group and let f be a complex-valued K-finite function on K such that $\int_K fn(k)dk = 0$ for any $n > 0$. Then for any K-finite function g, we have $\int_K fn(k)g(k)dk = 0$ for n large enough.

Take for instance the circle. In that case Mathieu's conjecture means :

For any Laurent polynomial f such that $c(f^n) = 0$ (constant term of) then for all Laurent polynomial g, $c(f^n g) = 0$ for all $n >> 0$.

Idea of the proof (Van der Kallen-Duistermaat)

Take the generating function :

$$\Sigma_n c^{n-1} \int_\gamma \frac{f(z)^n dz}{z} \tag{26}$$

and its analytic extension :

$$F(c) = \frac{1}{2\pi} \int_\gamma \frac{f(z)}{1 - cf(z)} \frac{dz}{z} \tag{27}$$

Because $f(0) = \infty$, $res(0) = -\frac{1}{c}$. As long as $\frac{1}{c}$ is not a critical value of $f(z)$, other residues are $-\frac{1}{c^2 f'(\xi)\xi}$ where $\xi = \xi_j(c)$ ranges over the solutions of $f(\xi) = \frac{1}{c}$. Assume that $f(0) = f(\infty) = \infty$ then the complex analytic extension of $\xi_j(c)$ can neither run to 0 or to ∞ when c remains bounded. If a is a critical point of f with critical value ν then $f(z)\ \nu + c(z-a)^m, m \geq 2$, $f'(z)\ cm(z-a)^{m-1}$, for $\nu \neq 0$ residue is of the order of $(\tau - \nu)^{-1+\frac{1}{m}}$. For

$\nu = 0$, res is of the order of $\tau^{1+\frac{1}{m}}$ which cannot cancel residue $-\tau$ at $z = 0$. Conclusion F cannt be identically zero.

Then either f is a polynomial in z or a polynomial in $1/z$. For such and f it is obvious that for all g, $c(f^n g) = 0$ for $n >> 0$.

Under the influence of Mathieu's conjecture Zhao developped a series of conjectures among which his "image conjecture".

Let A be the ring of polynomials and a_i a regular sequence. Let $\delta_i = \frac{d}{dz_i} - a_i : A \to A$ and $\delta = (\delta_1, ...\delta_n) : A^n \to A$ defined by $(\phi_1, ...\phi_n) \mapsto \Sigma\delta_i\phi_i$. Then for $f, g \in A$, if $f^k \in Im(\delta)$ for all k, then $f^k g \in Im(\delta)$ for $k >> 0$. The image conjecture implies the vanishing conjecture which is equivalent to the Jacobian conjecture.

III- The factorial conjecture

Van den essen, Wright and Zhao pointed out finally this year the following factorial conjecture as being fundamental in relation with the jacobian conjecture :

Let $D_n = \{(x_1, ..., x_n), x_i \geq 0\}$ a (complex) polynomial f so that

$$\int_{D_n} f(x)^k \exp(-\mathrm{x}_1 - ... - \mathrm{x}_\mathrm{n}) dx = 0 \tag{28}$$

for all k is necessarily $f = 0$. Hence moments are back but this time in any dimension.

This is a strong motivation to study moment problem in any dimensions (cf. JPF, F. Pakovich, Y. Yomdin and W. Zhao, Moment vanishing problem and positivity : some examples, Bull. Sci. Math. 135 (2011) 1, 10–32).

Thank you for your attention

Joyeux Anniversaire, Jaume !