Some new results on Darboux integrable differential systems

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Introduction

A *planar polynomial differential system* X is

$$\dot{x} = P(x,y),\ \dot{y} = Q(x,y), \qquad (1)$$

where $P, Q \in \mathbb{C}[x,y]$ are coprime.

The *degree* of X is $d = \max\{\deg P, \deg Q\}$.

A *first integral* of X is a non-constant $\mathcal{C}^1$-function H such that $PH_x + QH_y = 0$.

An *inverse integrating factor* of X is a $\mathcal{C}^1$-function V satisfying $PV_x + QV_y = \operatorname{div}(X)V$.

The inverse integrating factor V is *associated* to the first integral H of system (1) in U if $(P,Q) = (-H_y, H_x)V$ in $U \setminus \{V=0\}$.

Let $f \in \mathbb{C}[x,y]$. The algebraic curve $f=0$ is *invariant* by system (1) if there exists $K \in \mathbb{C}[x,y]$ of degree at most $d-1$ (the *cofactor*) such that $Pf_x + Qf_y = Kf$.

The function $F = e^{g/h}$ is an *exponential factor* of system (1) if there exists $L \in \mathbb{C}[x,y]$ of degree at most $d-1$ (the *cofactor*) such that $PF_x + QF_y = LF$. In this case, $h=0$ is invariant.

A Darboux function H can be written as

$$H(x,y) = \prod_{i=1}^{p} f_i^{\lambda_i} \exp\left(\frac{g}{\prod_{i=1}^{p} f_i^{n_i}}\right), \qquad (2)$$

where $f_i \in \mathbb{C}[x,y]$ is irreducible, $\lambda_i \in \mathbb{C}$, $n_i \in \mathbb{N}\cup\{0\}$ and $g \in \mathbb{C}[x,y]$ is coprime with f_i if $n_i > 0$, for all $i = 1,\dots,p$. Not both λ_i and n_i are zero for any i.

If H in (2) is a Darboux first integral of system (1) then the system has the rational inverse integrating factor

$$V(x,y) = \frac{\prod_{i=1}^{p} f_i^{n_i+1}}{R}, \qquad (3)$$

where $R \in \mathbb{C}[x,y]$ is the so-called *remarkable factor* (see [4]). It is the inverse integrating factor *associated* to the first integral $\log H$. If H is not rational then V is the only rational inverse integrating factor of system (1), see [1].

We define the extension of the degree function as the function δ given by

$$\delta\left(\prod g_i^{\alpha_i}\right) = \sum \alpha_i \deg g_i,$$

where $\alpha_i \in \mathbb{C}$ and $g_i \in \mathbb{C}[x,y]$.

It is known that if system (1) has a polynomial inverse integrating factor, then in the most of the cases it has degree $d+1$. In this work we compare $\delta(V)$ with $d+1$ in the rational case and we show under which hypotheses they are equal.

We also deal with the infinity, relating the singular points at infinity with all the polynomials appearing in (2), including g.

Still concerning Darboux integrable systems, we define remarkable values and remarkable curves for Darboux first integrals. They were first defined by Poincaré in [5] for rational first integrals. Their importance in the phase portrait of the system has been widely shown, see for example [1,3,4]. We extend these definitions to Darboux first integrals and prove a result that characterizes the existence of a polynomial inverse integrating factor by means of the number of critical remarkable values.

The infinity and the inverse integrating factor

THEOREM 1. Consider system (1) and suppose that it has a Darboux first integral (2) which is not rational and the inverse integrating factor V given in (3). The following statements hold.

(a) $\delta(V) < d+1$ if and only if $\delta(\Pi_2) > 0$. Moreover $\mathcal{F} = \delta(\Pi_2)\tilde{g}\prod_{i=1}^{p}\tilde{f}_i/\tilde{R} \not\equiv 0$.

(b) $\delta(V) = d+1$ if and only if either $\delta(\Pi_2) < 0$ and Π_1 is not constant, or $\delta(\Pi_2) = 0$. Moreover $\mathcal{F} = \delta(\Pi_1)\tilde{V}$.

(c) $\delta(V) > d+1$ if and only if $\delta(\Pi_2) < 0$ and Π_1 is constant. Moreover $\mathcal{F} \not\equiv 0$.

Corollary. The infinity is degenerate if and only if $\delta(\Pi_1) = 0$ and either $\delta(\Pi_2) < 0$ and Π_1 is not constant, or $\delta(\Pi_2) = 0$.

Corollary. If system (1) has a rational first integral H, then $\delta(V) = d+1$ and $\mathcal{F} = \delta(H)\tilde{V}$.

Remark. If R is constant, $g \equiv 0$ and $n_i = 0$ for all i, we have a generalization of a result of [2,6].

Remark. If R is constant, statement (e) of the Main Theorem of [1] assures that $\deg V = d+1$ under certain conditions. This statement is not always true. The correct hypotheses are the ones of Theorem 1.

Remarkable vaues and remarkable curves

The *remarkable values* are defined as level sets c of a minimal rational first integral $H = h_1/h_2$ for which $h_1 + ch_2$ factorizes into polynomials of lower degree. The factors of $h_1 + ch_2$ provide the *remarkable curves* associated to c. If in this factorization some factor has exponent greater than one, then the corresponding remarkable curve and remarkable value are said to be *critical*.

Suppose that system (1) has a Darboux first integral (2) which is not rational. Let $f=0$ be an irreducible invariant algebraic curve of system (1). We say that $f=0$ is a *critical remarkable curve* of H if either $f = f_i$ and $n_i > 0$, for some $i \in \{1,\dots,p\}$, or $f|R$. In the second case we say that $c = H|_{f=0} \in \mathbb{C}\setminus\{0\}$ is a *critical remarkable value* and we define the *exponent* of $f=0$ as its exponent in the factorization of R plus one.

We define the *exponent* of $f_i = 0$ as $n_i + 1$. We do not associate critical remarkable values to the f_i.

The remarkable factor R is formed by invariant algebraic curves different from the f_i. But the curves f_i and the ones appearing in the factorization of R are not the unique invariant algebraic curves that system (1) can have. We call these curves *non-critical remarkable curves* and the corresponding level sets of H *non-critical remarkable values*. We define their exponent as 1.

The polynomials f_i such that $n_i = 0$ are also considered non-critical remarkable curves, again without an associated remarkable value.

Remarkable values of Darboux first integrals

It is widely known that the inverse integrating factor usually belongs to an "smaller" class than the associated first integral:

(i) If H is Liouvillian, then V is Darboux.

(ii) If H is Darboux, then V is rational.

(iii) If H is rational, V is polynomial if and only if H has at most two critical remarkable values.

(iv) If H is polynomial, then V is polynomial.

THEOREM 2. If system (1) has a Darboux non-rational first integral (2), then V in (3) is a polynomial if and only if the number of critical remarkable values of H is zero.

Example

Let $H = y^2/(x+x^2+y^2)\exp(x^2(1+x)/y^2)$ and $V = y^3(x+x^2+y^2)$. From Theorem 1 $\mathcal{F} = x^3y(x^2+y^2)$, hence $d=5$. From Theorem 2 H has no critical remarkable curves. Moreover $x=0$ is a non-critical remarkable curve, as $x \neq f_i$, $x \nmid R$ and $H|_{x=0} = 1$.

Notations

We denote by $\tilde{f}$ the homogeneous part of highest degree of $f \in \mathbb{C}[x,y]$ and by $\mathcal{F} = x\tilde{Q} - y\tilde{P}$ the *characteristic polynomial* of X. We define

$$\Pi_1 = \prod_{i=1}^{p} \tilde{f}_i^{\lambda_i}, \quad \Pi_2 = \frac{\tilde{g}}{\prod_{i=1}^{p}\tilde{f}_i^{n_i}}.$$

Some references

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See the whole work in JMAA 394 (2012) 416–424.