Global Dynamics of the Lev Ginzburg Equation

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Introduction

In this work we study the global dynamics of the first order planar polynomial differential system of degree 2, called Lev Ginzburg differential system,

$$\begin{aligned} x' &= \frac{dx}{dt} = y, \\ y' &= \frac{dy}{dt} = (1-\beta_1 y)(\gamma - \alpha x + \beta y), \end{aligned} \tag{1}$$

depending on four parameters: $\alpha > 0$, $\beta_1 > 0$, $\gamma > 0$ and $\beta \in \mathbb{R}$. System (1) can be obtained from the following family of second order differential equations

$$\frac{d^2x}{dt^2} + \alpha\left(1 - \beta_1\frac{dx}{dt}\right)x = \left(1 - \beta_1\frac{dx}{dt}\right)\left(\gamma + \beta\frac{dx}{dt}\right), \tag{2}$$

introduced by Ginzburg [1] in his studies on population dynamics.

Bellamy and Mickens [5] claimed that the Lev Ginzburg differential equation (2) can exhibit a limit cycle coming from a Hopf bifurcation. In [2] the authors shown that this differential equation has neither a Hopf bifurcation, nor limit cycles.

Denote by $\mathcal{X} : \mathbb{R}^2 \to \mathbb{R}^2$ the quadratic vector field associated with the differential system (1), that is

$$\mathcal{X}(x,y) = (y, (1-\beta_1 y)(\gamma - \alpha x + \beta y)). \tag{3}$$

Differential system (1) presents only one equilibrium point $p = (\gamma/\alpha, 0)$ for all values of the parameters. The linearization $D\mathcal{X}(p)$ of $\mathcal{X}$ at p when $\beta = 0$ has eigenvalues $\lambda_{1,2} = \pm i\sqrt{\alpha}$. Then the equilibrium point p is either a center or a weak focus (see [3] for more details), and the standard Hopf bifurcation analysis can only be applied when the equilibrium is a weak focus, but the equilibrium point p when $\beta = 0$ is a center, see either [2] or item 1 of Theorem 1 below. The main result of this paper is the following.

Theorem 1. *Consider the Lev Ginzburg differential system* (1). *The following statements hold.*

1. If $\beta = 0$ then the only equilibrium point is a center.

2. If $\beta \neq 0$ then system (1) *has no limit cycles.*

In order to prove Theorem 1 we give full descriptions of the global behavior of system (1) for all values of the parameters.

Proof of Theorem 1, item 1

The following result is well known for the quadratic systems.

Proposition 2. *Let $\mathcal{X}$ be a quadratic vector field and let γ be a periodic orbit of $\mathcal{X}$. Then there is exactly one equilibrium point in the interior of γ. This equilibrium point is a center if γ is not a limit cycle, or it is a focus with complex conjugate eigenvalues when γ is a limit cycle.*

It is easy to check that the eigenvalues of $D\mathcal{X}(p)$ are given by $(\beta \pm \sqrt{\beta^2 - 4\alpha})/2$. Since we are interested in the limit cycles of the Lev Ginzburg differential system, by Proposition 2 we only need to consider such a differential system when the equilibrium point p can be a focus, that is when $\beta^2 < 4\alpha$, see [3] for more details. So, when $\beta \neq 0$ the equilibrium p is hyperbolic and its stability is directly determined by the sign of β, that is if $\beta > 0$ then p is an unstable focus, and if $\beta < 0$ then p is a stable focus. Now consider $\beta = 0$. By a translation, a linear change of variables and a rescaling of the independent variable t, system (1) can be written as

$$\dot{u} = -v + \beta_1 uv, \quad \dot{v} = u. \tag{4}$$

The linearization of (4) at the equilibrium localized at the origin has eigenvalues $\pm i$. In order to prove Proposition 4 we need the following result.

Theorem 3 (Bautin's Theorem [3]). *] Any quadratic system candidate to have a center at the origin of coordinates can be written in the normal form*

$$\begin{aligned} \dot{x} &= -y - \lambda_3 x^2 + (2\lambda_2 + \lambda_5)xy + \lambda_6 y^2, \\ \dot{y} &= x + \lambda_2 x^2 + (2\lambda_3 + \lambda_4)xy - \lambda_2 y^2. \end{aligned} \tag{5}$$

This equilibrium point localized at the origin is a center if and only if one of the following four conditions holds:

1. $\lambda_3 = \lambda_6$,

2. $\lambda_2 = \lambda_5 = 0$,

3. $\lambda_4 = \lambda_5 = 0$,

4. $\lambda_5 = \lambda_4 + 5(\lambda_3 - \lambda_6) = \lambda_3\lambda_6 - \lambda_2^2 - 2\lambda_6^2 = 0$.

Proposition 4. *The quadratic system* (4) *has a center at the origin.*

Demonstração. It is immediate that system (4) is in the form (5) and that $\lambda_3 = \lambda_6 = 0$. So, singularity point $(0,0)$ is a center. □

The proof of item 1 of Theorem 1 follows from Proposition 4.

Some classical results

Theorem 5. *If a quadratic polynomial differential system in the plane has an invariant straight line, then it has at most one limit cycle. Moreover, if this limit cycle exists then it is hyperbolic.*

Note that the Lev Ginzburg system (1) has the invariant straight line $y = 1/\beta_1$. So, by Theorem 5 system (1) has at most one limit cycle, and if it exists then it is hyperbolic.

The second result which also plays a main role for proving item 2 of Theorem 1 is the Poincaré compactification. In order to study the complete behavior of the trajectories of a planar polynomial differential system, we must study their behavior near infinity. Since we need this compactification for proving Theorem 1 in what follows we introduce it briefly. Let $\mathcal{X}$ be any planar vector field of degree n. The Poincaré compactified vector field $p(\mathcal{X})$ corresponding to $\mathcal{X}$ is an analytic vector field induced on $\mathbb{S}^2$ as follows. Let $\mathbb{S}^2 = \{y = (y_1, y_2, y_3) \in \mathbb{R}^3 : y_1^2 + y_2^2 + y_3^2 = 1\}$ (the Poincaré sphere) and $T_y\mathbb{S}^2$ be the tangent space to $\mathbb{S}^2$ at point y. Consider the central projection $f : T_{(0,0,1)}\mathbb{S}^2 \to \mathbb{S}^2$. This map defines two copies of $\mathcal{X}$, one in the northern hemisphere and the other in the southern hemisphere. Denote by $\mathcal{X}'$ the vector field $Df \circ \mathcal{X}$ defined on $\mathbb{S}^2$ except on its equator $\mathbb{S}^1 = \{y \in \mathbb{S}^2 : y_3 = 0\}$. Clearly $\mathbb{S}^1$ is identified to the infinity of $\mathbb{R}^2$. In order to extend $\mathcal{X}'$ to a vector field on $\mathbb{S}^2$ (including $\mathbb{S}^1$) it is necessary that $\mathcal{X}$ satisfies suitable conditions. In the case that $\mathcal{X}$ is polynomial $p(\mathcal{X})$ is the only analytic extension of $y_3^{n-1}\mathcal{X}'$ to $\mathbb{S}^2$. On $\mathbb{S}^2 \backslash \mathbb{S}^1$ there are two symmetric copies of $\mathcal{X}$, and knowing the behavior of $p(\mathcal{X})$ around $\mathbb{S}^1$, we know the behavior of $\mathcal{X}$ at infinity. The projection of the closed northern hemisphere of $\mathbb{S}^2$ on $y_3 = 0$ under $(y_1, y_2, y_3) \longmapsto (y_1, y_2)$ is called the Poincaré disc, and it is denoted by $\mathbb{D}^2$. The Poincaré compactification has the property that $\mathbb{S}^1$ is invariant under the flow of $p(\mathcal{X})$.

As $\mathbb{S}^2$ is a differentiable manifold, then for computing the expression for $p(\mathcal{X})$, we can consider the six local charts $U_i = \{y \in \mathbb{S}^2 : y_i > 0\}$, and $V_i = \{y \in \mathbb{S}^2 : y_i < 0\}$ where $i = 1, 2, 3$; and the diffeomorphisms $F_i : U_i \to \mathbb{R}^2$ and $G_i : V_i \to \mathbb{R}^2$ for $i = 1, 2, 3$ are the inverses of the central projections from the planes tangent at the points $(1,0,0), (-1,0,0), (0,1,0), (0,-1,0), (0,0,1)$ and $(0,0,-1)$ respectively. If we denote by $z = (z_1, z_2)$ the value of $F_i(y)$ or $G_i(y)$ for any $i = 1, 2, 3$ (so z represents different things according to the local charts under consideration), then some easy computations give the following expressions for $p(\mathcal{X})$:

$$z_2^n\Delta(z)\left(Q\left(\frac{1}{z_2},\frac{z_1}{z_2}\right) - z_1 P\left(\frac{1}{z_2},\frac{z_1}{z_2}\right), -z_2 P\left(\frac{1}{z_2},\frac{z_1}{z_2}\right)\right) \text{ in } U_1, \tag{6}$$

$$z_2^n\Delta(z)\left(P\left(\frac{z_1}{z_2},\frac{1}{z_2}\right) - z_1 Q\left(\frac{z_1}{z_2},\frac{1}{z_2}\right), -z_2 Q\left(\frac{z_1}{z_2},\frac{1}{z_2}\right)\right) \text{ in } U_2, \tag{7}$$

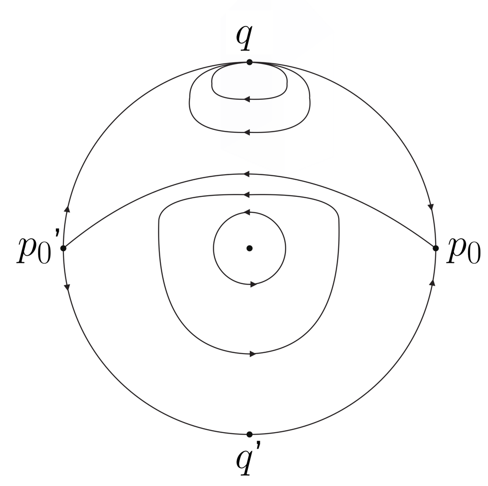
$$\Delta(z)\left(P(z_1,z_2), Q(z_1,z_2)\right) \quad \text{in } U_3,$$

where $\Delta(z) = (z_1^2 + z_2^2 + 1)^{-\frac{1}{2}(n-1)}$. The expression for V_i is the same as that for U_i except for a multiplicative factor $(-1)^{n-1}$. In these coordinates for $i = 1, 2$, $z_2 = 0$ always denotes the points of $\mathbb{S}^1$. In what follows we omit the factor $\Delta(z)$ by rescaling the vector field $p(\mathcal{X})$. Thus we obtain a polynomial vector field in each local chart of degree at most $n+1$. Note that since the equilibrium points at infinity are of the form $(z_1, 0)$, at infinity there are at most $n+1$ pairs of equilibrium points, because if we have an equilibrium point the symmetric point with respect to the center of the Poincaré sphere is also another equilibrium point. With the local chart U_1 and its symmetric V_1 we cover all the infinity except the origin of the local charts U_2 and V_2. So, using the symmetry with respect to the origin of the Poincaré sphere for studying the infinity we only need to study the chart U_1 and the origin of the chart U_2. Moreover, since our polynomial differential system is quadratic if, for instance, we have a stable node in the chart U_1 then its symmetric in the chart V_1 is an unstable node.

Proof of Theorem 1, item 2

We start by studying the infinity of the Lev Ginzburg differential system (1). Doing a translation of the unique equilibrium point of system (1) at the origin, and writing its linear part in its real Jordan normal form the Lev Ginzburg system becomes

$$\begin{aligned} x' = P(x,y) &= \frac{\beta}{2}x - \frac{\sqrt{4\alpha - \beta^2}}{2}y - \frac{\beta\beta_1}{2}xy - \frac{\beta^2\beta_1}{2\sqrt{4\alpha - \beta^2}}y^2, \\ y' = Q(x,y) &= \frac{\sqrt{4\alpha - \beta^2}}{2}x + \frac{\beta}{2}y - \frac{\sqrt{4\alpha - \beta^2}\beta_1}{2}xy - \frac{\beta\beta_1}{2}y^2. \end{aligned} \tag{8}$$



The phase portrait of system (8) with $\beta = 0$.

Now we write the polynomial differential system (8) in the local chart U_1 using (6) and we obtain

$$\begin{aligned} z_1' &= \frac{\sqrt{4\alpha - \beta^2}}{2}(-\beta_1 z_1 + z_2 + z_1^2 z_2) + \frac{\beta^2\beta_1}{2\sqrt{4\alpha - \beta^2}}z_1^3, \\ z_2' &= \frac{\beta\beta_1}{2}z_1 z_2 - \frac{\beta}{2}z_2^2 + \frac{\beta^2\beta_1}{2\sqrt{4\alpha - \beta^2}}z_1^2 z_2 + \frac{\sqrt{4\alpha - \beta^2}}{2}z_1 z_2^2. \end{aligned}$$

Therefore the equilibrium points $(z_1, 0)$ at infinity in the local chart U_1 are

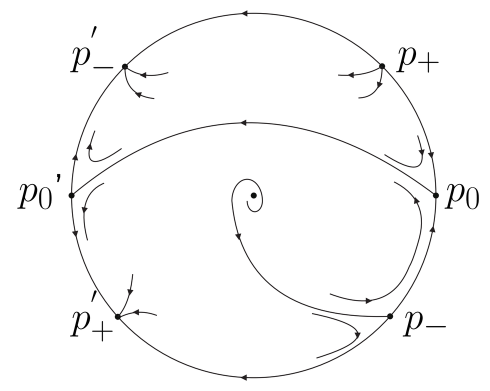
$$p_0 = (0,0), \quad p_\pm = \left(\pm\frac{\sqrt{4\alpha - \beta^2}}{\beta}, 0\right), \quad \text{when } \beta \neq 0,$$

and only the equilibrium point p_0 when $\beta = 0$. The symmetric points with respect to the origin of the Poincaré disc in the local chart V_1 are denoted by p_0' and $p_\pm'$. Clearly if $\beta \neq 0$, then all the infinite equilibrium points are in the local charts U_1 and V_1 having the maximum of possible equilibrium points at infinity, that is three pairs of diametrally opposite equilibrium points in the Poincaré disc.

When $\beta = 0$ then in U_1 we only have the equilibrium p_0, and consequently in V_1 we only have the equilibrium p_0'. In this case we will see that the origins of the local charts U_2 and V_2 are also equilibrium points at infinity. Writing the polynomial differential system (8) in the local chart U_2 using (7) we get

$$\begin{aligned} z_1' &= \sqrt{\alpha}(-z_2 + \beta_1 z_1^2 - z_1^2 z_2), \\ z_2' &= \sqrt{\alpha}(\beta_1 z_1 z_2 - z_1 z_2^2). \end{aligned}$$

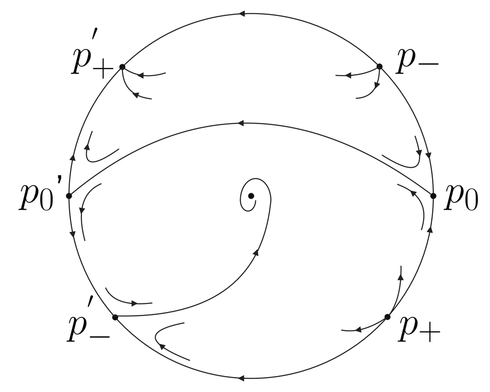
Therefore the origin of the local chart U_2 is a nilpotent equilibrium point, by using Theorem 3.5, we obtain that it is formed by an elliptic and a hyperbolic sectors with the elliptic sector contained in U_2 and the hyperbolic sector contained in V_2. By Proposition 4 we know that the origin is a center. So, the phase portrait for $\beta = 0$ is topologically equivalent to the one of Figure 3.



The phase portrait of system (8) with $4\alpha - \beta^2 > 0$ and $\beta > 0$.

Now assume that $\beta \neq 0$. Then using Theorem 2.15 (for the hyperbolic equilibria) and Theorem 2.19 (for the semi–hyperbolic equilibria) of [6] we obtain that p_0 is a semi–hyperbolic saddle, p_+ is a hyperbolic unstable node, and p_- is a semi–hyperbolic saddle–node which in U_1 has the two hyperbolic sectors when $\beta > 0$ and the node sector when $\beta < 0$. Note that looking at Figures 1 and 2 in the half–plane under the invariant straight line $y = 1/\beta_1$ the origin and the corresponding node at infinity have converse kind of stability. Hence limit cycles cannot surround the origin, because at most can have one hyperbolic limit cycle, and consequently the origin and the node at infinity would have the same kind of stability. In summary, when $\beta \neq 0$ the phase portraits of system (8) are topologically equivalent to the ones of Figures 1 and 2 without limit cycles. This completes the proof of item 2 of Theorem 1.

The proof of item 1 of Theorem 1 follows from Proposition 4.



The phase portrait of system (8) with $4\alpha - \beta^2 > 0$ and $\beta < 0$

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