Relaxation oscillations in slow-fast systems

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New Trends in Dynamical Systems

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In this survey talk we study systems depending on two time scales (slow-fast systems), using geometric techniques (by "geometric" we mean coming from " regular" dynamical systems theory). We limit to two-dimensional systems, however depending on an arbitrary number of parameters. Our attention primarily goes to periodic orbits, called relaxation oscillations, and the bifurcations that they undergo.

The talk is based on joint work with Robert Roussarie, with Peter De Maesschalck or with both.

The results hold on any smooth orientable surface (without boundary) M.

Let $X_{\varepsilon,\lambda}$ be a smooth family of vector fields on M, defined for $\varepsilon \in [0, \varepsilon_1]$ (for a given $\varepsilon_1 > 0$) and for $\lambda \in \Lambda$, with Λ a subset of an euclidean space.

We assume that $X_{\varepsilon,\lambda}$ is of slow-fast type, with singular parameter ε:
-for $\varepsilon = 0, X_{0,\lambda}$ can locally be written as $F(x, y, \lambda)\partial/\partial x$
-there exists a smooth λ-family of 1-dimensional **embedded** manifolds S_λ consisting entirely of singularities of $X_{0,\lambda}$.

We do not require S_λ to be connected; in fact one could have several connected components (curves).

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Definition 1 (Normally hyperbolic point and contact point)

A point $p \in S_\lambda$ is called a *normally hyperbolic* (resp. normally attracting or normally repelling) point of $X_{0,\lambda}$ if the linear part of $X_{0,\lambda}$ at p has a nonzero (resp. negative or positive) eigenvalue.
It is called a *contact point* when the linear part has two zero eigenvalues.
In that case, we distinguish between a nilpotent and a degenerate contact point, depending on whether the differential of $X_{\varepsilon,\lambda}$ at p is nilpotent or is zero.

We will only treat slow-fast cycles with isolated nilpotent contact points. We denote the set of contact points by C_λ.
Near nilpotent singularities, the set of singularities of $X_{0,\lambda}$ forms a regular curve.

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Proposition 1

Consider a smooth slow-fast system $X_{\varepsilon,\lambda}$ on a smooth surface M. Let p be a nilpotent contact point for a parameter value $\lambda = \lambda_0$. There exist smooth local coordinates (x, y) such that $p = (0, 0)$, and in which, up to multiplication by a smooth strictly positive function, the system $X_{\varepsilon,\lambda}$, for $(\varepsilon, \lambda) \sim (0, \lambda_0)$, is written in the following normal form:

$$\left\{ \begin{array}{rcl} \dot{x} & = & y - f(x, \lambda) \\ \dot{y} & = & \varepsilon\Big(g(x, \varepsilon, \lambda) + \big(y - f(x, \lambda)\big)h(x, y, \varepsilon, \lambda)\Big), \end{array} \right. \tag{1}$$

for smooth functions f, g, h and $f(0, \lambda_0) = \frac{\partial f}{\partial x}(0, \lambda_0) = 0$.

This means that $X_{\varepsilon,\lambda}$ is C^∞-equivalent to the normal form (1).

Remarks:

- Near a normally hyperbolic point there exists following local normal form for C^∞-equivalence:

$$\left\{\begin{array}{rcl} \dot{x} & = & y - x \\ \dot{y} & = & \varepsilon\Big(g(x,\varepsilon,\lambda) + \big(y-x\big)h(x,y,\varepsilon,\lambda)\Big), \end{array}\right. \tag{2}$$

- We will not treat problems concerning the time function. We can hence work with local normal forms for C^∞-equivalence.

Expression (1) is a specification of the usual local expression of a slow-fast system in *fast time.*

$$\left\{ \begin{array}{rcl} \dot{x} & = & F(x,y,\varepsilon,\lambda) \\ \dot{y} & = & \varepsilon G(x,y,\varepsilon,\lambda). \end{array} \right. \tag{3}$$

For $\varepsilon = 0$ we have the layer equation

$$\left\{ \begin{array}{rcl} \dot{x} & = & F(x,y,0,\lambda) \\ \dot{y} & = & 0. \end{array} \right. \tag{4}$$

Consider a system written in the normal form (1). Outside the set of contact points $C_\lambda = \{(x,y) \mid \frac{\partial f_\lambda}{\partial x}(x) = 0, y = f_\lambda(x)\}$ the slow dynamics is defined on the slow curve S_λ by the equation

$$\frac{\partial f_\lambda}{\partial x}(x)\dot{x} = g(x,\lambda).$$

The zeros of g_λ on $S_\lambda \setminus C_\lambda$ are the zeros of the slow dynamics with as order the order of the zero of g_λ.
This order has an intrinsic meaning, independent of the choice of the normal form. This also holds at contact points, allowing the following definition of invariants for a contact point:

Definition 2 (Order at contact point)

Consider any normal form (1) near a contact point p of a slow-fast system $X_{\varepsilon,\lambda}$ for the value λ_0. The order $\mathrm{Ord}|_0(f_{\lambda_0})$, (≥ 2), is called the *contact order*. The order $\mathrm{Ord}|_0(g_{0,\lambda_0})$, (≥ 0), is called the *singularity order*. The contact point p is said to be *regular* if the singularity order is zero (i.e. if $g_{0,\lambda_0}(0) \neq 0$) and *singular* if not.

Remark: The singularity order represents the algebraic multiplicity of the (isolated) singularities that we will encounter near $(0,0)$ for $\varepsilon > 0$.

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Definition 3

Consider any normal form (1) near a singular contact point p of singularity order 1. The point is said to be a singular contact point of index ± 1 when

$$\operatorname{sign} \frac{\partial g_{0,\lambda_0}}{\partial x}(0) = \mp 1.$$

For such a singular contact point, the family of vector fields $X_{\varepsilon,\lambda}$ has, for (ε, λ) close to $(0, \lambda_0)$, a singular point $(x, y) = (x_0(\varepsilon, \lambda)), y_0(\varepsilon, \lambda))$ tending to $(0, 0)$ as $(\varepsilon, \lambda) \to (0, \lambda_0)$.

This singularity is non-degenerate for $\varepsilon > 0$ sufficiently small. It is of saddle type when it is a singularity of index -1, and of center/focus type when it is a singularity of index $+1$.

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Definition 4 (Specification on slow-fast cycles)

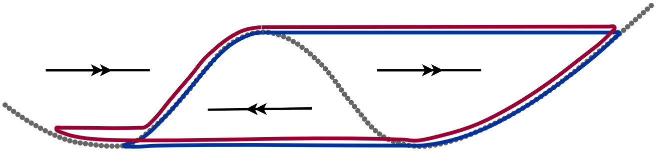
Given $\lambda_0 \in \Lambda$. A subset Γ, diffeomorphic to a piecewise smooth circle consisting of a finite number of fast orbits of X_{0,λ_0} and a finite number of slow arcs of X_{0,λ_0}, is called a *slow-fast cycle* of X_{0,λ_0} if it contains at least one slow arc.
Moreover, it must be possible to orient the circle in a way that the orientation is compatible to the orientation on the fast orbits and such that on all slow arcs it agrees with the orientation of the slow dynamics.

A slow-fast cycle is said to be a *regular* slow-fast cycle if there are no singularities for the slow dynamics on the slow arcs and if each contact point is regular.

Definition 5

A *common slow-fast cycle* is a slow-fast cycle for which the slow arcs are either all attracting or all repelling. A *canard cycle* is a slow-fast cycle that is not common.

A slow-fast cycle is called a *strongly common slow-fast cycle* if it is a **regular** common slow-fast cycle that is not approached by nearby canard (slow-fast) cycles.(All contact points are supposed to be nilpotent of finite order.)



Theorem 1 (on existence)

Let Γ be a **strongly common** *slow-fast cycle of $X_{\varepsilon,\lambda_0}$ for a given $\lambda_0 \in \Lambda$. Then there exists an $\varepsilon_0 > 0$, a neighborhood $\Lambda_0 \subset \Lambda$ of λ_0 and a neighborhood $\mathcal{T}$ of Γ such that for $\varepsilon \in]0, \varepsilon_0]$ and $\lambda \in \Lambda_0$, the vector field $X_{\varepsilon,\lambda}$ has a limit cycle in $\mathcal{T}$. This limit cycle is unique and hyperbolic, and tends in Hausdorff sense towards Γ as $(\varepsilon, \lambda) \to (0, \lambda_0)$.*

The statement is not true for arbitrary regular common slow-fast cycles: the presence of nearby canard slow-fast cycles may create the possibility of having no nearby limit cycles for certain parameter values.

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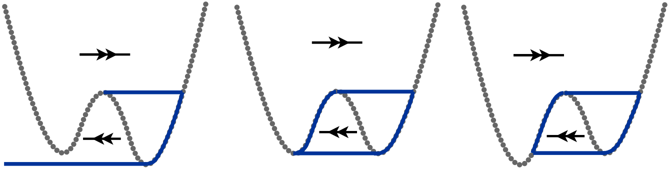


$\Longrightarrow$ for cycles that are close to canard cycles, one can in general not guarantee the existence of nearby periodic orbits.

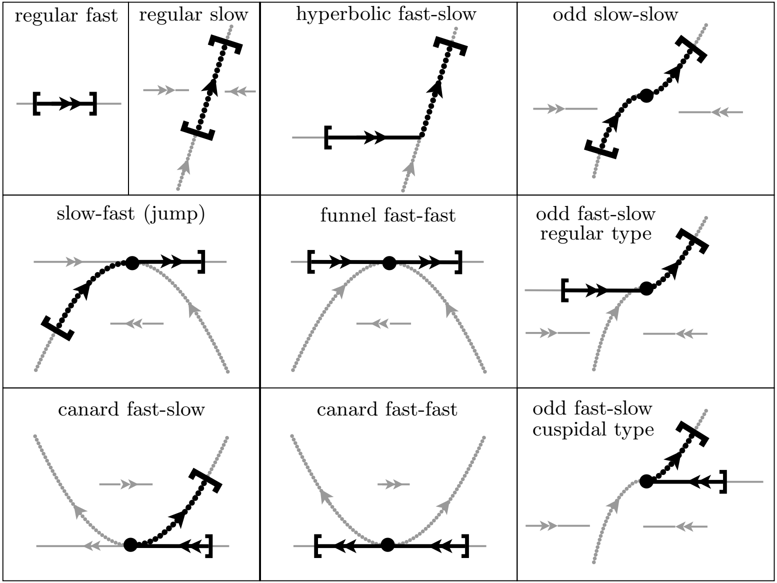


Figure: Elementary attracting regular slow-fast segments.

Proof of the (first part of the) Theorem: we make a covering with "slow-fast flow boxes"

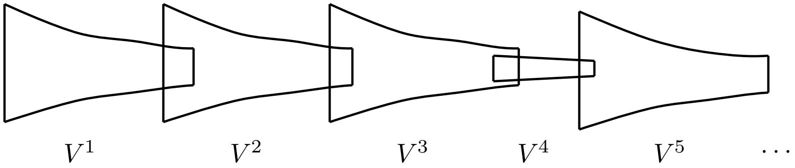


Figure: Chain of flow box neighborhoods near some segments.

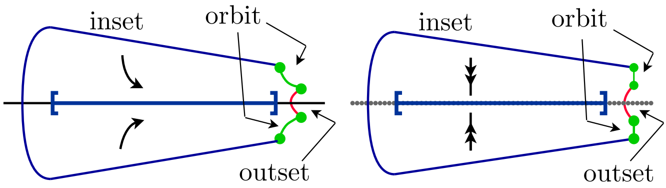


Figure: Flow box neighborhood near a regular slow segment. To the left for $\varepsilon > 0$; the limit as $\varepsilon \to 0$ to the right.

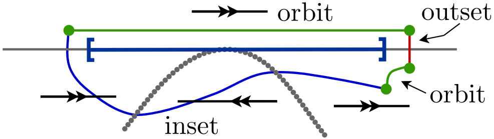


Figure: Flow box neighborhood near a funnel fast–fast segment.

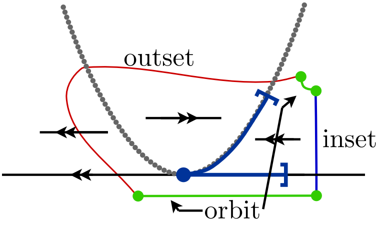
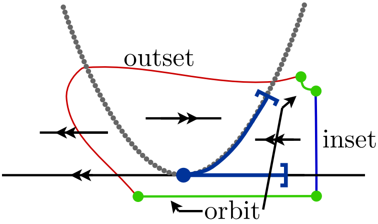


Figure: Flow box neighborhood near a canard fast–slow segment (left) and a canard fast–fast segment (right)

When we consider slow-fast cycles with singularities on the slow arcs, or with singular nilpotent contact points, one cannot expect to have slow-fast families of flow box neighborhoods. Instead, we will have to work with other types of *well-adapted neighborhoods*.



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What about the stability of a relaxation oscillation near Γ? How many periodic orbits are there near Γ?

Known property: given a planar vector field X and a closed orbit γ. Then γ is an attracting cycle when

$$\int_\gamma divX\,dt < 0. \tag{5}$$

In a slow-fast context: divergence integrals along slow arcs contribute most. Associated to slow arcs, we define the notion *slow divergence integral*:

$$I(p,q,\lambda) = \int_p^q divX_{0,\lambda}\,ds, \tag{6}$$

where we integrate along the slow arc of the slow curve from p to q w.r.t. the so-called slow time s, i.e. the time induced by the slow dynamics.

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Near regular normally hyperbolic points and regular contact points, the relation between the divergence integral and the slow divergence integral is clear:

$$\left\{ \begin{array}{rcl} \dot{x} & = & y - f(x,\lambda) \\ \dot{y} & = & \varepsilon\Big(g(x,\varepsilon,\lambda) + (y - f(x,\lambda))h(x,y,\varepsilon,\lambda)\Big). \end{array} \right. \tag{7}$$

So

$$divX_{\varepsilon,\lambda} = -\frac{\partial f}{\partial x} + O(\varepsilon) \tag{8}$$

and

$$dt = \frac{dy}{\dot{y}} = \frac{1}{\varepsilon}\frac{dy}{g(x,0,\lambda) + o(1)} \tag{9}$$

This implies

$$\int divX_{\varepsilon,\lambda}\,dt = \frac{1}{\varepsilon}\left(\int divX_{0,\lambda}ds + o(1)\right) \tag{10}$$

Theorem 2 (on unicity)

Let Γ *be a slow-fast cycle of* $X_{\varepsilon,\lambda_0}$ *on* M *for given* $\lambda_0 \in \Lambda$. *We suppose:*

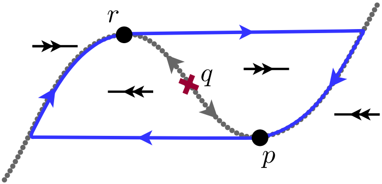
1. *All the contact points are nilpotent and of finite order. They are regular contact points, or of singularity index* $+1$.
2. *If* Γ *does not contain singularities of the slow dynamics, then*

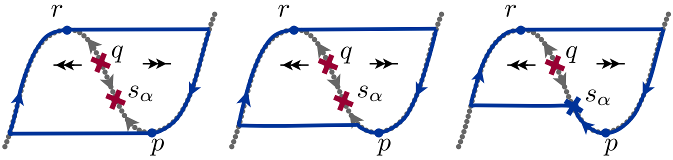
$$\int_\Gamma \operatorname{div} X_{0,\lambda_0}\, ds \neq 0.$$

3. *If* Γ *contains singularities of the slow dynamics (outside contact points), they are all located on hyperbolic arcs of the same type (all attracting or all repelling).*

Under assumptions (1)–(3) there exists a $\delta > 0$, *an* $\varepsilon_0 > 0$ *and a neighborhood* Λ_0 *of* λ_0 *such that for any* $(\varepsilon, \lambda) \in]0, \varepsilon_0] \times \Lambda_0$, *there is at most one closed orbit of* $X_{\varepsilon,\lambda}$ *which is* δ*-Hausdorff close to* Γ. *If it appears, this closed orbit is a hyperbolic limit cycle, attracting (resp. repelling) if the slow divergence integral is negative (resp. positive).*

$$X_{\varepsilon,b}\colon \begin{cases} \dot{x} &=& y - x^2 - x^3 \\ \dot{y} &=& \varepsilon(b_0 + b_1 x - x^2 - 3x^3) \end{cases} \tag{11}$$



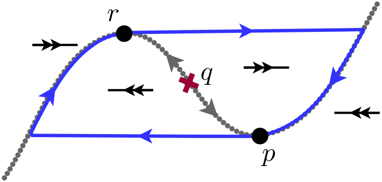


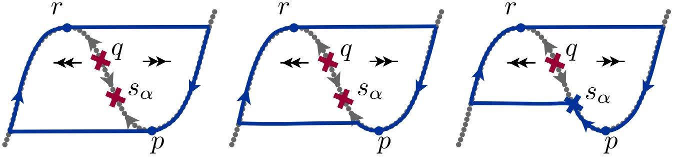
(a) (b) (c)

Proposition 2

The above slow-fast cycle Γ can, for (ε, b_0, b_1) close to $(0,0,0)$, be approached by generic saddle-node bifurcations of limit cycles.

$$X_{\varepsilon,b}: \begin{cases} \dot{x} &= y - x^2 - x^3 \\ \dot{y} &= \varepsilon(b_0 + b_1 x - x^2 - 3x^3) \end{cases} \qquad (11)$$





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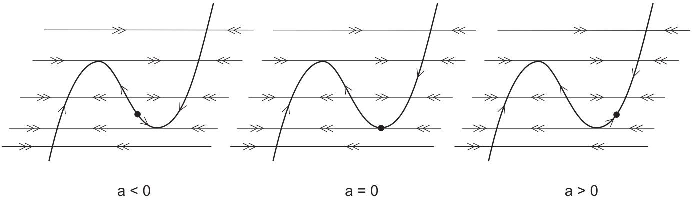
Hopf breaking mechanism

Example: Van der Pol's equation with Hopf bifurcation:

$$\left\{ \begin{array}{lcl} \dot{x} & = & y - \frac{x^2}{2} - \frac{x^3}{3} \\ \dot{y} & = & \varepsilon(a - x). \end{array} \right. \tag{12}$$

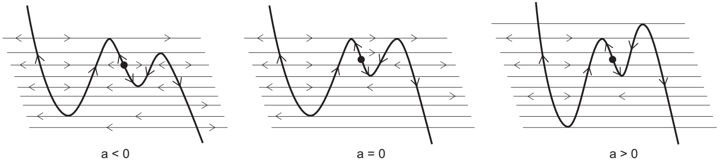
The critical curve $\{y = \frac{x^2}{2} + \frac{x^3}{3}\}$ does not depend on a, but the slow dynamics does: $x(1+x)x' = a - x$



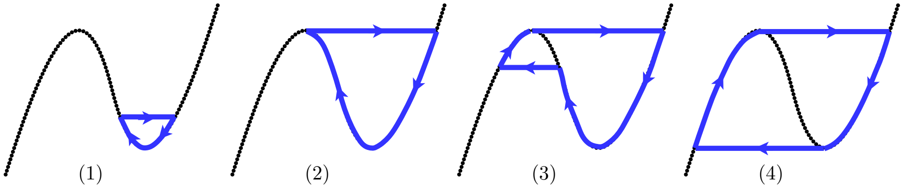
Jump breaking mechanism

Example: A Liénard equation with generic crossing of two maxima:

$$\begin{cases} \dot{x} = y - 3 + (x^2 - 1)^2(x + 3) - ax \\ \dot{y} = -\varepsilon x. \end{cases} \tag{13}$$



Canard cycles



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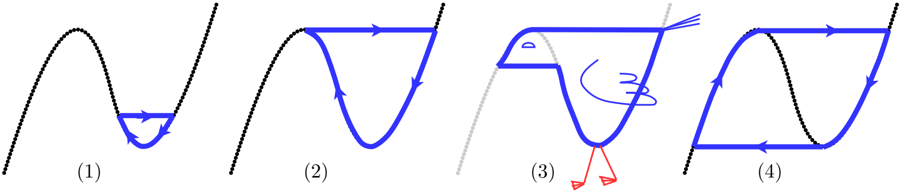

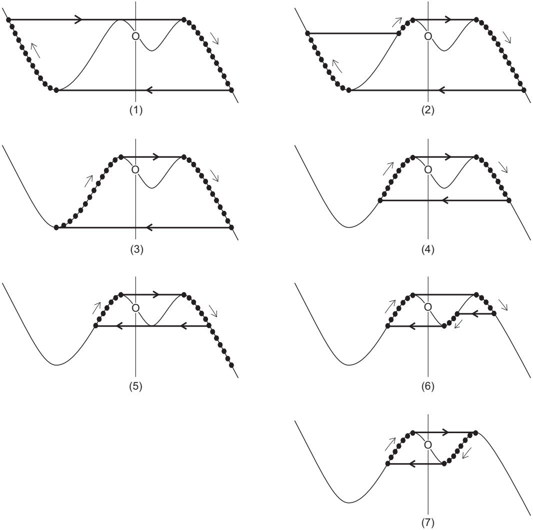
O
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Generic canard cycles in generic breaking mechanism

Theorem 3

If $I(Y,\mu)$ has at $Y = Y_0$ (corresponding to Γ_0) and $\mu = \mu_0$, a zero of multiplicity n together with a full unfolding of it (i.e. a catastrophe of codimension $n-1$), then for $\varepsilon > 0$ sufficiently small, the (a,μ)-family $X_{\varepsilon,a,\mu}$ contains near Γ_0, and for $(a,\mu) \sim (0,\mu_0)$ a limit cycle of multiplicity $n+1$ together with a full unfolding of it.

On the proof of the theorems 2 and 3

How to study regular orbits near slow curves ?

A result of Fenichel [Fe] describes the dynamics near compact pieces of normally hyperbolic slow curves. It is based on the center manifold reduction and use of appropriate normal forms. However, instead of presenting Fenichel's result, we will rely on a theorem of Takens [Ta], which permits a stronger result, for sure in this 2-dimensional context.

Let $X_{\varepsilon,\lambda}$ be a C^∞ family of vector fields on a 2-dimensional manifold M. Locally we think about

$$\left\{ \begin{array}{rcl} \dot{x} & = & f(x,y,\varepsilon,\lambda) \\ \dot{y} & = & \varepsilon g(x,y,\varepsilon,\lambda). \end{array} \right. \tag{14}$$

It reveals to be interesting to see ε as a variable, rather than as a parameter, i.e. we consider $X_{\varepsilon,\lambda} + 0\frac{\partial}{\partial\varepsilon}$ as a C^∞-family of vector fields on $M \times [0,\varepsilon_0[$, with $\varepsilon_0 > 0$. For the equation (14) it means that we add "$\dot{\varepsilon} = 0$".

We also consider the critical curve, that we denote by γ_λ, for each λ, as a curve in three dimensions: $\gamma_\lambda \subset M \times \{0\} \subset M \times [0,\varepsilon_0[$.

The Normal Linearization Theorem of Takens shows that X, near a normally hyperbolic point p of Γ, is, for any k, C^k- equivalent to:

$$\begin{cases} \dot{v} = \pm v \\ \dot{u} = \varepsilon F(u, \varepsilon, \lambda). \end{cases} \tag{15}$$

where $\{v = 0\}$ stands for an a priori chosen center manifold.
Let us continue with the case $\dot{v} = -v$.

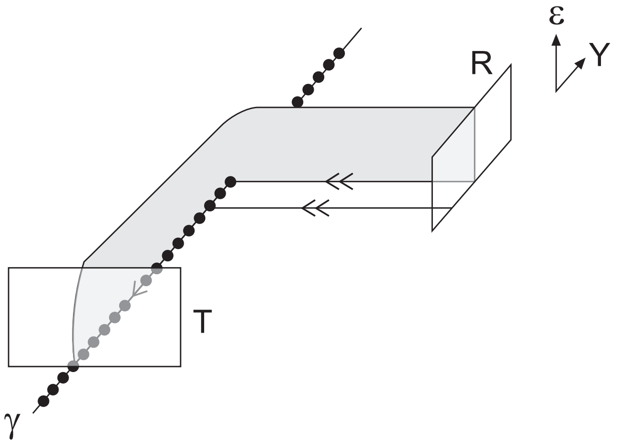
If we suppose that the slow dynamics on Γ is non-zero at the point p for $\lambda = \lambda_0$; i.e. $F(0, 0, \lambda_0) \neq 0$, then using the Flow Box Theorem, we can change the coordinate u, with a coordinate change depending in a C^k way on (ε, λ), such that expression (15) changes into:

$$\begin{cases} \dot{v} = -v \\ \dot{u} = \varepsilon. \end{cases} \tag{16}$$

In these coordinates the solution with initial conditions (u_0, v_0) is given by:

$$v = v_0 \; exp\left(-\frac{1}{\varepsilon}(u - u_0)\right).$$

Seen in 3-space (u, v, ε), and for each λ separately, we get a behaviour of the solutions as represented in the following figure. The shaded surface is smooth, except at the corner point.



Solutions near a normally hyperbolic slow curve.

Study at contact points

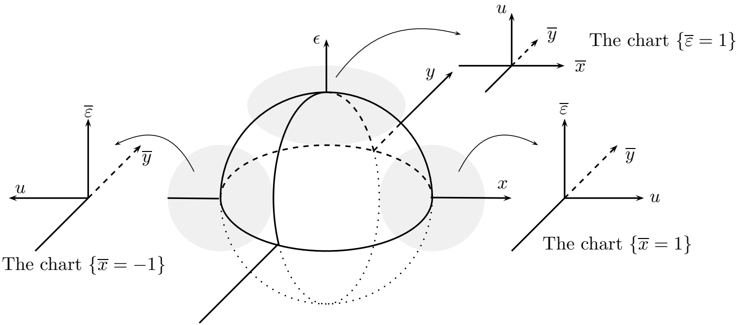
To study the behaviour of regular $X_{\varepsilon,\lambda}$-orbits, with $\varepsilon > 0$, near a contact point we use blow up (geometric desingularization). We present it for a single vector field in (x, y, ε)-space. Let us suppose that the contact point is situated at $(x, y, \varepsilon) = (0, 0, 0)$ for all $\lambda \in \Lambda$.

Blow up procedure

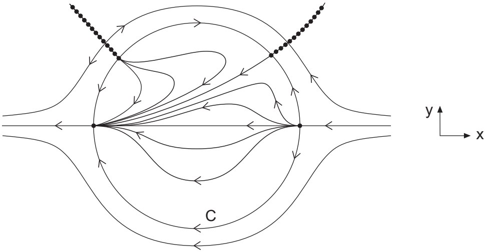
If we have a family of 3-dimensional vector fields $X_{\varepsilon,\lambda} + 0\frac{\partial}{\partial\varepsilon}$, and we want to blow up the origin $(x, y, \varepsilon) = (0, 0, 0)$, then we use

$$\begin{cases} x = u^p \overline{x} \\ y = u^q \overline{y} \\ \varepsilon = u^m \overline{\varepsilon}, \end{cases} \tag{17}$$

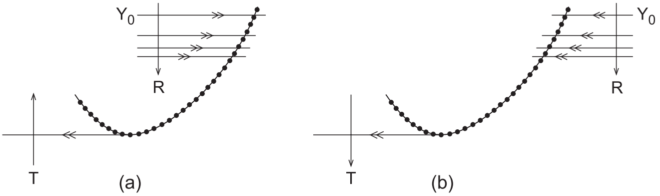
with $(\overline{x}, \overline{y}, \overline{\varepsilon}) \in S^2$, and $u \in [0, \infty[$; we choose a new time $\overline{t} = u^r t$. The coefficients (p, q, m, r) are natural numbers, to be well chosen.



Blowing up a singular point.



Blown up picture of a jump point.



Sections in the jump mechanism.

Definition 6 (ε-regularly smoothness)

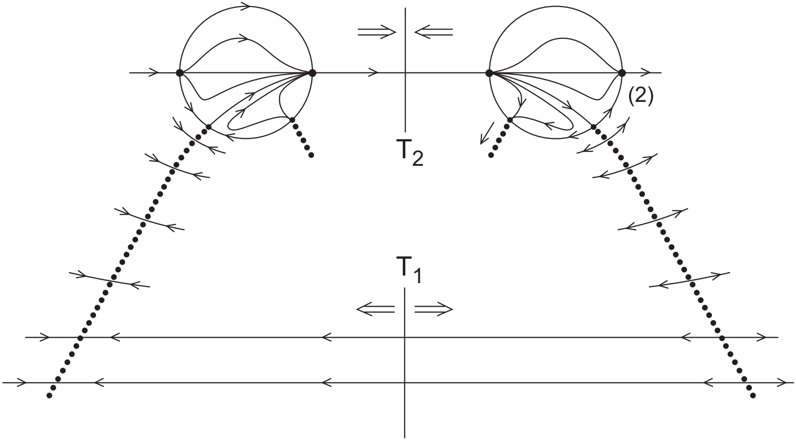
We say that a function $f(z, \varepsilon)$, with $z \in \mathbb{R}^p$, for some p, is *ε-regularly smooth in z* (or ε-regularly C^∞ in z) if f is continuous and all partial derivatives of f with respect to z exist and are continuous in (z, ε), including at $\varepsilon = 0$.

Theorem 4

If we follow the X-orbits from R to T and denote this transition map as P, then P can be expressed as

$$P(Y,\varepsilon,\lambda) = (\alpha_0(\varepsilon,\lambda) + exp(\frac{1}{\varepsilon}(\alpha(Y,\varepsilon,\lambda))),\varepsilon), \tag{18}$$

where both α_0 and α are ε-regularly smooth in respectively λ and (Y,λ).
The graph of $\{U = \alpha_0(\varepsilon,\lambda)\}$ represents $W_{\Sigma_0} \cap T$, and
$\alpha(Y,0,\lambda) = I(Y,\lambda)$, the slow divergence integral.
The function exp $(\frac{1}{\varepsilon}(\alpha(Y,\varepsilon,\lambda))$ is smooth in (Y,ε,λ).



We choose transverse sections T_1 and T_2 and C^∞ coordinates (Y, ε) and (U, ε) on respectively T_1 and T_2.

We denote the transition from T_1 to T_2 in forward time by Δ_1 and the transition in backward time by Δ_2.
The limit cycles, for $\varepsilon > 0$, $\varepsilon \sim 0$, and $\lambda \sim \lambda_0$, correspond to solutions of

$$\Delta_1(Y, \varepsilon, \lambda) = \Delta_2(Y, \varepsilon, \lambda).$$

We know that $\Delta_i(Y, \varepsilon, \lambda) = (D_i(Y, \varepsilon, \lambda), \varepsilon)$ with

$$D_i(Y, \varepsilon, \lambda) = f_i(\varepsilon, \lambda) + exp(\frac{1}{\varepsilon}(A_i(Y, \varepsilon, \lambda))), \tag{19}$$

where the functions f_i and A_i are ε-regularly smooth in respectively λ and (Y, λ).

The limit cycles of X_ε, for $\varepsilon > 0$, correspond to solutions of

$$f_1(\varepsilon,\lambda) - f_2(\varepsilon,\lambda) + exp(\frac{1}{\varepsilon}(A_1(Y,\varepsilon,\lambda))) - exp(\frac{1}{\varepsilon}(A_2(Y,\varepsilon,\lambda))) = 0. \quad (20)$$

Hopf breaking mechanism

By a smooth equivalence the family of equations can locally be written as

$$\begin{cases} \dot{x} &= y - \left(x^2 + \sum_{i=3}^{n} a_i(\lambda)x^i + x^{n+1}k(x,\lambda)\right) \\ \dot{y} &= -\varepsilon(b(\lambda) + x + yc(x,y,\varepsilon,\lambda)). \end{cases} \tag{21}$$

It is part of the universal unfolding

$$\begin{cases} \dot{x} &= y - \left(x^2 + \sum_{i=3}^{n} a_i x^i + x^{n+1}k(x,\lambda)\right) \\ \dot{y} &= -\varepsilon(b + x + yc(x,y,\varepsilon,\lambda)). \end{cases} \tag{22}$$

The interesting parameter region is given by

$$(\varepsilon, b) = (\tau^2, \tau B),$$

with $\tau \geq 0$.

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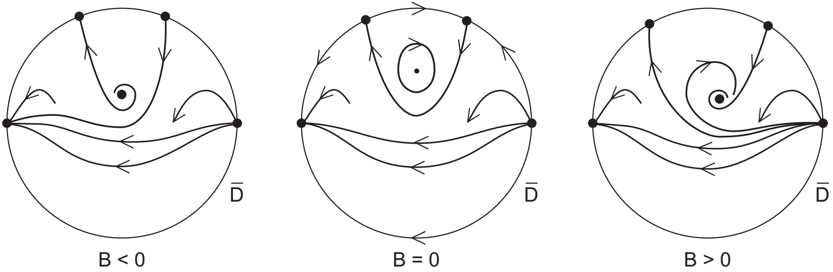
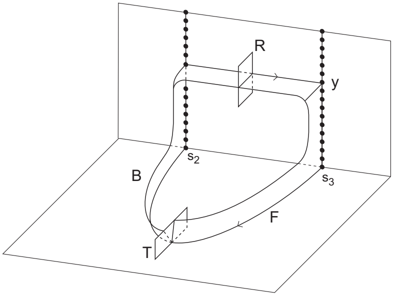
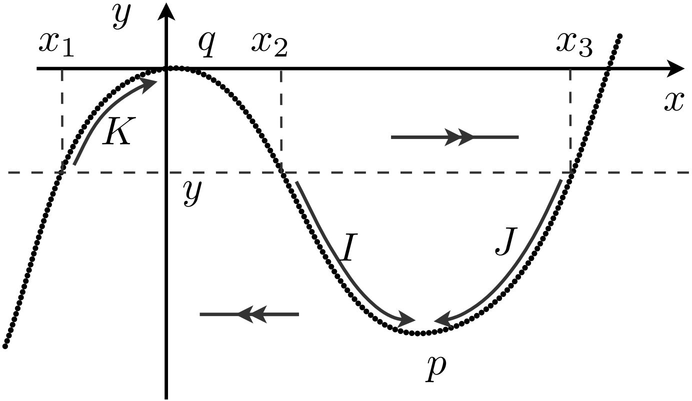


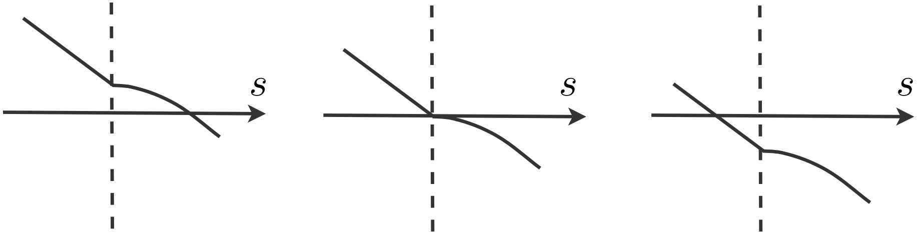
Blown-up pictures in Hopf breaking mechanism.



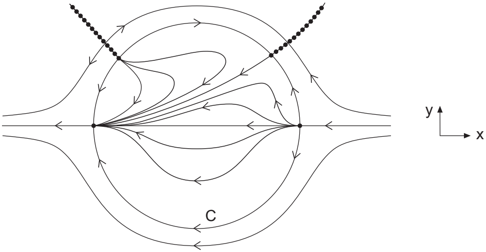
Three-dimensional picture of mappings $\mathcal{F}$ and $\mathcal{B}$.

Transitory canard in Hopf breaking mechanism





$J_0(\lambda) > I_0(\lambda)$ $\qquad$ $J_0(\lambda) = I_0(\lambda)$ $\qquad$ $J_0(\lambda) < I_0(\lambda)$



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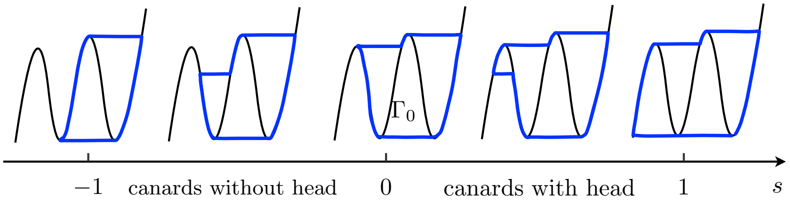
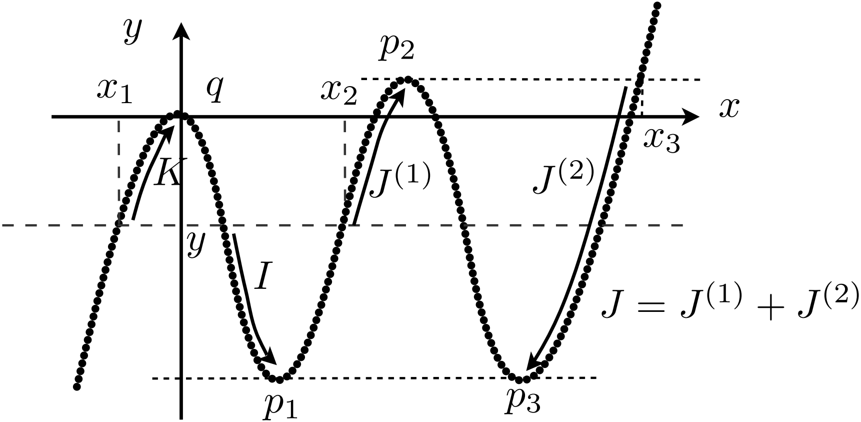
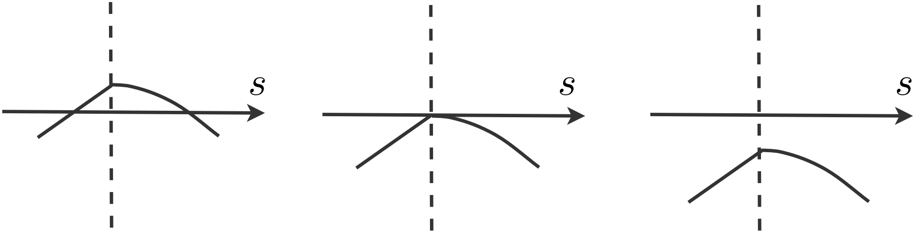




y
p_2
x_1
q
x_2
x
x_3
K
$J^{(1)}$
$J^{(2)}$
y
I
$J = J^{(1)} + J^{(2)}$
p_1
p_3



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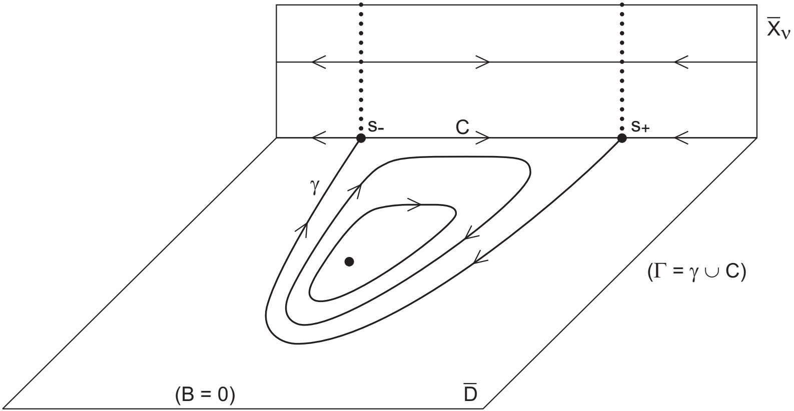
Birth of canard cycles in Hopf breaking mechanism



Along the blow-up locus limit cycles are only possible for $B = 0$, hence near

$$\begin{cases} \dot{\bar{x}} = \bar{y} - \bar{x}^2 \\ \dot{\bar{y}} = -\bar{x}, \end{cases} \tag{23}$$

This vector field represents a time-reversible center, having

$$H(\bar{x}, \bar{y}) = e^{-2\bar{y}}(\bar{y} - \bar{x}^2 + \frac{1}{2})$$

as first integral and $-2e^{-2\bar{y}}$ as integrating factor. The only singularity of (23) is situated at $(0,0)$ with $H(0,0) = \frac{1}{2}$. The system has an invariant parabola given by

$$\gamma = \{\bar{y} = \bar{x}^2 - \frac{1}{2}\},$$

where $H \equiv 0$; there is a unique nest of invariant ovals γ_h with $h \in]0, \frac{1}{2}[$.

The limit cycles will be found near $(\bar{x}, \bar{y}) = (0, 0)$, near some γ_h with $h \in]0, \frac{1}{2}[$, or they will come close to γ.

The study near the Hopf point situated at $(\bar{x}, \bar{y}) = (0, 0)$ is easy.

Finding precise upper bounds on the number of limit cycles near ovals γ_h, with h restricted to a compact interval in $]0, \frac{1}{2}[$, relies on the study of the integrals

$$J_{2j+1}(h) = 2 \int_{\gamma_h} e^{-2\bar{y}} \bar{x}^{2j+1} d\bar{y}, \quad j = 0, 1, 2, \dots,$$

with γ_h oriented counter-clockwise.

To get precise upper bounds on the number of limit cycles near a slow-fast Hopf point it is better not to work with $\{J_{2j+1}(h)\}$ itself, but with the derivatives

$$\frac{d}{dh}(J_{2j+1})(h) = (2j+1)\bar{J}_{2j+1} \quad \text{where} \quad \bar{J}_{2j+1} = \int_{\gamma_h} \bar{x}^{2j-1} d\bar{y}. \qquad (24)$$

SCS-Conjecture *For each* $q \geq 0$, *the functions* $\bar{J}_{2j+1} \quad j = 0, 1, 2, \ldots, q$ *form a strict Chebyshev system on intervals* $[h_0, \frac{1}{2}]$ *for each* $h_0,\ 0 < h_0 < \frac{1}{2}$.

Theorem 5

Jaume Llibre did a great job till now.

Proof:

See MathSciNet (among other things).

Conjecture:

He will continue doing so.

Thankyou!

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