Rotopulsating orbits of the curved n-body problem

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Goal: to present some results from

- F. Diacu. On the singularities of the curved n-body problem, *Trans. Amer. Math. Soc.* **363**, 4 (2011), 2249-2264.
- F. Diacu and E. Pérez-Chavela. Homographic solutions of the curved 3-body problem, *J. Differential Equations* **250** (2011), 340-366.
- F. Diacu. Polygonal homographic orbits of the curved n-body problem, *Trans. Amer. Math. Soc.* **364**, 5 (2012), 2783-2802.
- F. Diacu. *Relative equilibria in the curved N-body problem*, Atlantis Studies in Dynamical Systems, vol. I, Atlantis Press, 2012.
- F. Diacu. Relative equilibria in the 3-dimensional curved n-body problem, *Memoirs Amer. Math. Soc.* (to appear).
- F. Diacu and Shima Kordlou. Rotopulsating orbits of the curved N-body problem (in progress).

Notations

Consider $m_1, \ldots, m_n > 0$ in $\mathbb{S}^3 \subset \mathbb{R}^4$ for positive Gaussian curvature and $\mathbb{H}^3 \subset \mathbb{M}^{3,1}$ (Minkowski space) for negative Gaussian curvature, where

$$\mathbb{S}^3 = \{(w, x, y, z) | w^2 + x^2 + y^2 + z^2 = 1\},$$

$$\mathbb{H}^3 = \{(w, x, y, z) | w^2 + x^2 + y^2 - z^2 = -1,\ z > 0\},$$

with positions given by $\mathbf{q}_i = (w_i, x_i, y_i, z_i),\ i = \overline{1, n}$.

$\mathbf{q} = (\mathbf{q}_1, \ldots, \mathbf{q}_n)$ configuration of the system

$\nabla_{\mathbf{q}_i} := (\partial_{w_i}, \partial_{x_i}, \partial_{y_i}, \sigma\partial_{z_i}), \nabla := (\nabla_{\mathbf{q}_1}, \ldots, \nabla_{\mathbf{q}_n})$ the gradient

$\sigma = \begin{cases} +1 & \text{in } \mathbb{S}^3 \\ -1 & \text{in } \mathbb{H}^3 \end{cases}$ the signum function

$$\mathbf{a} := (a_w, a_x, a_y, a_z), \mathbf{b} := (b_w, b_x, b_y, b_z),$$

$\mathbf{a} \cdot \mathbf{b} := (a_w b_w + a_x b_x + a_y b_y + \sigma a_z b_z)$ the inner product

Equations of motion

In general:

$$\boxed{\ddot{\mathbf{q}}_i = \sum_{j=1, j\neq i}^{n} \frac{m_j[\mathbf{q}_j - \sigma(\mathbf{q}_i \cdot \mathbf{q}_j)\mathbf{q}_i]}{[\sigma - \sigma(\mathbf{q}_i \cdot \mathbf{q}_j)^2]^{3/2}} - \sigma(\dot{\mathbf{q}}_i \cdot \dot{\mathbf{q}}_i)\mathbf{q}_i,}$$

$$\mathbf{q}_i \cdot \mathbf{q_i} = \sigma, \quad \mathbf{q}_i \cdot \dot{\mathbf{q}}_i = 0, \quad i = \overline{1, n}$$

$$\mathbb{S}^3: \qquad \ddot{\mathbf{q}}_i = \sum_{j=1, j\neq i}^{n} \frac{m_j[\mathbf{q}_j - (\mathbf{q}_i \cdot \mathbf{q}_j)\mathbf{q}_i]}{[1 - (\mathbf{q}_i \cdot \mathbf{q}_j)^2]^{3/2}} - (\dot{\mathbf{q}}_i \cdot \dot{\mathbf{q}}_i)\mathbf{q}_i,$$

$$\mathbf{q}_i \cdot \mathbf{q_i} = 1, \quad \mathbf{q}_i \cdot \dot{\mathbf{q}}_i = 0, \quad i = \overline{1, n}$$

$$\mathbb{H}^3: \qquad \ddot{\mathbf{q}}_i = \sum_{j=1, j\neq i}^{n} \frac{m_j[\mathbf{q}_j + (\mathbf{q}_i \cdot \mathbf{q}_j)\mathbf{q}_i]}{[(\mathbf{q}_i \cdot \mathbf{q}_j)^2 - 1]^{3/2}} + (\dot{\mathbf{q}}_i \cdot \dot{\mathbf{q}}_i)\mathbf{q}_i,$$

$$\mathbf{q}_i \cdot \mathbf{q_i} = -1, \quad \mathbf{q}_i \cdot \dot{\mathbf{q}}_i = 0, \quad i = \overline{1, n}$$

Hamiltonian form and energy integral

$$\mathbf{p} := (\mathbf{p}_1, \dots, \mathbf{p}_n), \ \ \mathbf{p}_i := m_i \dot{\mathbf{q}}_i, \ \ i = \overline{1, n},$$ momenta

$$T(\mathbf{q}, \mathbf{p}) = \tfrac{1}{2} \textstyle\sum_{i=1}^{n} m_i^{-1} (\mathbf{p}_i \cdot \mathbf{p}_i)(\sigma \mathbf{q}_i \cdot \mathbf{q}_i)$$ kinetic energy

$$H(\mathbf{q}, \mathbf{p}) = T(\mathbf{q}, \mathbf{p}) - U(\mathbf{q})$$ Hamiltonian function

$$\begin{cases} \dot{\mathbf{q}}_i = \nabla_{\mathbf{p}_i} H(\mathbf{q}, \mathbf{p}) = m_i^{-1} \mathbf{p}_i, \\ \dot{\mathbf{p}}_i = -\nabla_{\mathbf{q}_i} H(\mathbf{q}, \mathbf{p}) = \nabla_{\mathbf{q}_i} U(\mathbf{q}) - \sigma m_i^{-1} (\mathbf{p}_i \cdot \mathbf{p}_i) \mathbf{q}_i, \\ \mathbf{q}_i \cdot \mathbf{q}_i = \sigma, \ \ \mathbf{q}_i \cdot \mathbf{p}_i = 0, \ \ i = \overline{1, n} \end{cases}$$

$$H(\mathbf{q}, \mathbf{p}) = h$$ energy integral

– there are no first integrals of the centre of mass and the linear momentum

Integrals of the total angular momentum

$$\sum_{i=1}^{n} m_i \mathbf{q}_i \wedge \dot{\mathbf{q}}_i = \mathbf{c},$$

$$\mathbf{c} = c_{wx}\mathbf{e}_w\wedge\mathbf{e}_x + c_{wy}\mathbf{e}_w\wedge\mathbf{e}_y + c_{wz}\mathbf{e}_w\wedge\mathbf{e}_z + c_{xy}\mathbf{e}_x\wedge\mathbf{e}_y + c_{xz}\mathbf{e}_x\wedge\mathbf{e}_z + c_{yz}\mathbf{e}_y\wedge\mathbf{e}_z,$$

$$\mathbf{e}_w = (1,0,0,0),\ \mathbf{e}_x = (0,1,0,0),\ \mathbf{e}_y = (0,0,1,0),\ \mathbf{e}_z = (0,0,0,1),$$

$$c_{wx}, c_{wy}, c_{wz}, c_{xy}, c_{xz}, c_{yz} \in \mathbb{R}.$$

On components, there are 6 integrals:

$$\sum_{i=1}^{n} m_i(w_i\dot{x}_i - \dot{w}_i x_i) = c_{wx}, \quad \sum_{i=1}^{n} m_i(w_i\dot{y}_i - \dot{w}_i y_i) = c_{wy},$$

$$\sum_{i=1}^{n} m_i(w_i\dot{z}_i - \dot{w}_i z_i) = c_{wz}, \quad \sum_{i=1}^{n} m_i(x_i\dot{y}_i - \dot{x}_i y_i) = c_{xy},$$

$$\sum_{i=1}^{n} m_i(x_i\dot{z}_i - \dot{x}_i z_i) = c_{xz}, \quad \sum_{i=1}^{n} m_i(y_i\dot{z}_i - \dot{y}_i z_i) = c_{yz}$$

Isometries in $\mathbb{S}^3$

The isometries of $\mathbb{S}^3$ are given by the Lie group $SO(4)$:

$$A = \begin{pmatrix} \cos\theta & -\sin\theta & 0 & 0 \\ \sin\theta & \cos\theta & 0 & 0 \\ 0 & 0 & \cos\phi & -\sin\phi \\ 0 & 0 & \sin\phi & \cos\phi \end{pmatrix}, \theta, \phi \in [0, 2\pi).$$

Isometries in $\mathbb{H}^3$

The isometries of $\mathbb{H}^3$ are given by the Lorentz group of $\mathbb{M}^{3,1}$:

$$B = \begin{pmatrix} \cos\theta & -\sin\theta & 0 & 0 \\ \sin\theta & \cos\theta & 0 & 0 \\ 0 & 0 & \cosh\phi & \sinh\phi \\ 0 & 0 & \sinh\phi & \cosh\phi \end{pmatrix}, \theta \in [0, 2\pi), \phi \in \mathbb{R},$$

$$C = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -\xi & \xi \\ 0 & \xi & 1-\xi^2/2 & \xi^2/2 \\ 0 & \xi & -\xi^2/2 & 1+\xi^2/2 \end{pmatrix}, \xi \in \mathbb{R}.$$

Positive elliptic RP orbits

A solution of the equations of motion in $\mathbb{S}^3$ is called a positive elliptic rotopulsating orbit if it is of the form

$$\mathbf{q} = (\mathbf{q}_1, \mathbf{q}_2, \ldots, \mathbf{q}_n), \quad \mathbf{q}_i = (w_i, x_i, y_i, z_i), \quad i = \overline{1, n},$$
$$w_i = r_i(t)\cos[\alpha(t) + a_i], x_i = r_i(t)\sin[\alpha(t) + a_i], y_i = y_i(t), z_i = z_i(t),$$

where $a_i,\ i = \overline{1, n}$, are constants, α is not a constant function, r_i, y_i, and z_i satisfy the conditions

$$0 \le r_i \le 1; \quad -1 \le y_i, z_i \le 1; \quad r_i^2 + y_i^2 + z_i^2 = 1, \quad i = \overline{1, n},$$

and $c_{yz} = 0$. If r is constant and $\alpha(t) = \bar{\alpha} t$, with $\bar{\alpha}$ a nonzero constant, then the solution is called a positive elliptic relative equilibrium.

Criterion for positive elliptic RP orbits

A solution candidate of the above form is a positive elliptic RP orbit if and only if $c_{yz} = 0, \dot\alpha = \frac{c}{\sum_{j=1}^n m_j(1-y_j^2-z_j^2)}$, where c is a constant, and the variables $y_i, z_i,\ i = \overline{1,n}$, satisfy the system of $2n$ second-order differential equations

$$\begin{cases} \ddot y_i = \sum_{\substack{j=1\\ j\neq i}}^n \frac{m_j(y_j - q_{ij}y_i)}{(1-q_{ij}^2)^{3/2}} - G_i y_i \\ \ddot z_i = \sum_{\substack{j=1\\ j\neq i}}^n \frac{m_j(z_j - q_{ij}z_i)}{(1-q_{ij}^2)^{3/2}} - G_i z_i, \end{cases}$$

where

$$G_i := \frac{\dot y_i^2 + \dot z_i^2 - (y_i\dot z_i - z_i \dot y_i)^2}{1 - y_i^2 - z_i^2} + \frac{c^2(1 - y_i^2 - z_i^2)}{\left[\sum_{j=1}^n m_j(1 - y_j^2 - z_j^2)\right]^2},$$

$i = \overline{1,n}$, and, for any $i, j \in \{1, 2, \dots, n\}$, q_{ij} is given by

$$q_{ij} := \mathbf{q_i} \cdot \mathbf{q_j} = (1 - y_i^2 - z_i^2)^{\frac12}(1 - y_j^2 - z_j^2)^{\frac12}\cos(a_i - a_j) + y_i y_j + z_i z_j.$$

Examples

Positive elliptic Lagrangian RP orbits

$$m_1 = m_2 = m_3 =: m > 0, \quad r^2 + \rho^2 = 1$$

$$\mathbf{q} = (\mathbf{q}_1, \mathbf{q}_2, \mathbf{q}_3), \quad \mathbf{q}_i = (w_i, x_i, y_i, z_i), \quad i = 1, 2, 3,$$

$$w_1 = r(t)\cos\alpha(t),\ x_1 = r(t)\sin\alpha(t),\ y_1 = y(t),\ z_1 = z(t),$$

$$w_2 = r(t)\cos\left[\alpha(t) + \frac{2\pi}{3}\right],\ x_2 = r(t)\sin\left[\alpha(t) + \frac{2\pi}{3}\right], y_2 = y(t),\ z_2 = z(t),$$

$$w_3 = r(t)\cos\left[\alpha(t) + \frac{4\pi}{3}\right],\ x_3 = r(t)\sin\left[\alpha(t) + \frac{4\pi}{3}\right], y_3 = y(t),\ z_3 = z(t).$$

The study of these orbits reduces to the system

$$\begin{cases} \dot{z} = v \\ \dot{v} = \left[\frac{2m(5-9\delta^2 z^4)}{\sqrt{3}(1-\delta z^2)^{\frac{1}{2}}(1+3\delta z^2)^{\frac{3}{2}}} - \frac{2h}{3m}\right] z, \end{cases}$$

where $\delta \geq 1$ is a constant and h is the energy constant. So for admissible initial conditions, existence and uniqueness of analytic solutions is assured.

Positive elliptic-elliptic RP orbits

A solution of the equations of motion in $\mathbb{S}^3$ is called a positive elliptic-elliptic rotopulsating orbit if it is of the form

$$\mathbf{q} = (\mathbf{q}_1, \mathbf{q}_2, \dots, \mathbf{q}_n),\ \mathbf{q}_i = (w_i, x_i, y_i, z_i),\ i = \overline{1, n},$$

$$w_i = r_i(t)\cos[\alpha(t) + a_i],\ x_i = r_i(t)\sin[\alpha(t) + a_i],$$

$$y_i = \rho_i(t)\cos[\beta(t) + b_i],\ z_i = \rho_i(t)\sin[\beta(t) + b_i],$$

where $a_i, b_i,\ i = \overline{1, n}$, are constants, α and β are not constant functions, and r_i and ρ_i satisfy the conditions

$$0 \le r_i, \rho_i \le 1 \ \text{ and } \ r_i^2 + \rho_i^2 = 1,\ i = \overline{1, n}.$$

When r and ρ are constant and $\alpha(t) = \bar{\alpha}t, \beta(t) = \bar{\beta}t$, with $\bar{\alpha}, \bar{\beta}$ nonzero constants, then the solution is called a positive elliptic-elliptic relative equilibrium.

Criterion for positive elliptic-elliptic RP orbits

If $M = \sum_{i=1}^n m_i$, asolution candidate of the above form is a rotopulsating positive elliptic-elliptic orbit if and only if

$$\dot\alpha = \frac{c_1}{\sum_{i=1}^n m_i r_i^2}, \quad \dot\beta = \frac{c_2}{M - \sum_{i=1}^n m_i r_i^2},$$

with c_1, c_2 constants, and the variables $r_1, r_2, \dots, r_n$ satisfy the n second-order differential equations

$$\ddot r_i = r_i(1-r_i^2)\left[\frac{c_1^2}{(\sum_{i=1}^n m_i r_i^2)^2} - \frac{c_2^2}{(M - \sum_{i=1}^n m_i r_i^2)^2}\right] - \frac{r_i \dot r_i^2}{1-r_i^2}$$
$$+ \sum_{\substack{j=1\\ j\ne i}}^n \frac{m_j[r_j(1-r_i^2)\cos(a_i - a_j) - r_i(1-r_i^2)^{\frac12}(1-r_j^2)^{\frac12}\cos(b_i - b_j)]}{(1-\epsilon_{ij}^2)^{\frac32}},$$

where, for any $i, j \in \{1, 2, \dots, n\}$ with $i \ne j$, we denoted

$$\epsilon_{ij} := \mathbf{q}_i \cdot \mathbf{q}_j = r_i r_j \cos(a_i - a_j) + (1-r_i^2)^{\frac12}(1-r_j^2)^{\frac12}\cos(b_i - b_j).$$

Examples

Positive elliptic-elliptic Lagrangian RP orbits

$$m_1 = m_2 = m_3 =: m,$$

$$\mathbf{q} = (\mathbf{q}_1, \mathbf{q}_2, \mathbf{q}_3), \quad \mathbf{q}_i = (w_i, x_i, y_i, z_i), \quad i = 1, 2, 3,$$

$$w_1 = r(t)\cos\alpha(t),\ x_1(t) = r(t)\sin\alpha(t),\ y_1 = \rho(t)\cos\beta(t),\ z_1(t) = \rho(t)\sin\beta(t),$$

$$w_2 = r(t)\cos[\alpha(t) + 2\pi/3],\ x_2(t) = r(t)\sin[\alpha(t) + 2\pi/3],$$

$$y_2 = \rho(t)\cos[\beta(t) + 2\pi/3],\ z_2(t) = \rho(t)\sin[\beta(t) + 2\pi/3],$$

$$w_3 = r(t)\cos[\alpha(t) + 4\pi/3],\ x_3(t) = r(t)\sin[\alpha(t) + 4\pi/3],$$

$$y_3 = \rho(t)\cos[\beta(t) + 4\pi/3],\ z_3(t) = \rho(t)\sin[\beta(t) + 4\pi/3],$$

with α and β nonconstant functions and $r^2 + \rho^2 = 1$.

It follows that

$$\dot{\alpha} = \frac{c_1}{3mr^2}, \quad \dot{\beta} = \frac{c_2}{3m(1-r^2)},$$

with $c_1 = c_{wx}$ and $c_2 = c_{yz}$, both nonzero, and the equations of motion reduce to the system

$$\begin{cases} \dot{r} = u, \\ \dot{u} = \frac{c_1^2(1-r^2)}{9m^2r^3} - \frac{r(9m^2u^2+c_2^2)}{9m^2(1-r^2)}. \end{cases}$$

For each admissible initial conditions, this system yields a unique analytic solution.

Remarkable fact: These orbits maintain the same size, but they cannot be generated by the action of any single element of the Lie group SO(4).

Negative elliptic RP orbits

A solution of the equations of motion in $\mathbb{H}^3$ is called a negative elliptic rotopulsating orbit if it is of the form

$$\mathbf{q} = (\mathbf{q}_1, \mathbf{q}_2, \dots, \mathbf{q}_n), \quad \mathbf{q}_i = (w_i, x_i, y_i, z_i), \quad i = \overline{1,n},$$
$$w_i = r_i(t)\cos[\alpha(t) + a_i], x_i = r_i(t)\sin[\alpha(t) + a_i], y_i = y_i(t), z_i = z_i(t),$$

where $a_i,\ i = \overline{1,n}$, are constants, α is not a constant function, r_i, y_i, and z_i satisfy the conditions

$$z_i \geq 1 \quad \text{and} \quad r_i^2 + y_i^2 - z_i^2 = -1, \quad i = \overline{1,n},$$

and $c_{yz} = 0$. If r is constant and $\alpha(t) = \bar{\alpha}t$, with $\bar{\alpha}$ a nonzero constant, then the solution is called a negative elliptic relative equilibrium.

Criterion for negative elliptic RP orbits

A solution candidate as above is a positive elliptic rotopulsating orbit for the equations of motion if and only if

$$\dot\alpha = \frac{b}{\sum_{j=1}^n m_j(z_j^2 - y_j^2 - 1)},$$

where b is a constant, and the variables $y_i, z_i,\ i = \overline{1,n}$, satisfy the system of $2n$ second-order differential equations

$$\begin{cases} \ddot y_i = \sum_{\substack{j=1\\ j\neq i}}^n \frac{m_j(y_j+\mu_{ij}y_i)}{(\mu_{ij}^2-1)^{3/2}} + F_i y_i \\ \ddot z_i = \sum_{\substack{j=1\\ j\neq i}}^n \frac{m_j(z_j+\mu_{ij}z_i)}{(\mu_{ij}^2-1)^{3/2}} + F_i z_i, \end{cases}$$

where

$$F_i := \frac{[(y_i\dot z_i - z_i\dot y_i)^2 + \dot z_i^2 - \dot y_i^2]}{z_i^2 - y_i^2 - 1} + \frac{b^2(z_i^2 - y_i^2 - 1)}{[\sum_{j=1}^n m_j(z_j^2 - y_j^2 - 1)]^2},$$

$i = \overline{1,n}$, and, for any $i, j \in \{1, 2, \dots, n\}$, μ_{ij} is given by

$$\mu_{ij} := \mathbf{q}_i \cdot \mathbf{q_j} = (z_i^2 - y_i^2 - 1)^{\frac{1}{2}}(z_j^2 - y_j^2 - 1)^{\frac{1}{2}}\cos(a_i - a_j) + y_i y_j - z_i z_j.$$

Examples

Negative elliptic Lagrangian RP orbits

$$m_1 = m_2 = m_3 =: m > 0, \quad r^2 + y^2 - z^2 = -1$$

$$\mathbf{q} = (\mathbf{q}_1, \mathbf{q}_2, \mathbf{q}_3), \quad \mathbf{q}_i = (w_i, x_i, y_i, z_i), \quad i = 1, 2, 3,$$

$$w_1 = r(t)\cos\alpha(t),\ x_1 = r(t)\sin\alpha(t),\ y_1 = y(t),\ z_1 = z(t),$$

$$w_2 = r(t)\cos\left[\alpha(t) + \frac{2\pi}{3}\right], x_2 = r(t)\sin\left[\alpha(t) + \frac{2\pi}{3}\right], y_2 = y(t), z_2 = z(t)$$

$$w_3 = r(t)\cos\left[\alpha(t) + \frac{4\pi}{3}\right], x_3 = r(t)\sin\left[\alpha(t) + \frac{4\pi}{3}\right], y_3 = y(t), z_3 = z(t)$$

The equations of motion reduce to the system

$$\begin{cases} \dot{z} = u \\ \dot{u} = \left[\frac{2h}{3m} - \frac{2m(5 - 9\epsilon^2 z^4)}{\sqrt{3}(\epsilon z^2 - 1)^{\frac{1}{2}}(3\epsilon z^2 + 1)^{\frac{3}{2}}}\right] z, \end{cases}$$

with $0 < \epsilon \leq 1$ constant, so existence and uniqueness follows.

Negative hyperbolic RP orbits

A solution of the equations of motion in $\mathbb{H}^3$ is called a negative hyperbolic rotopulsating orbit if it is of the form

$$\mathbf{q} = (\mathbf{q}_1, \mathbf{q}_2, \dots, \mathbf{q}_n), \quad \mathbf{q}_i = (w_i, x_i, y_i, z_i), \quad i = \overline{1, n},$$
$$w_i = w_i(t), \quad x_i = x_i(t),$$
$$y_i = \eta_i(t) \sinh[\beta(t) + b_i], \quad z_i = \eta_i(t) \cosh[\beta(t) + b_i],$$

where $b_i,\ i = \overline{1, n}$, are constants, β is not a constant function, w_i, x_i, z_i, and η_i satisfy the conditions

$$z_i \geq 1 \quad \text{and} \quad w_i^2 + x_i^2 - \eta_i^2 = -1, \quad i = \overline{1, n},$$

and $c_{wx} = 0$. If η is constant and $\beta(t) = \bar{\beta} t$, with $\bar{\alpha}$ a nonzero constant, then the solution is called a negative hyperbolic relative equilibrium.

Criterion for negative hyperbolic RP orbits

A solution candidate as above is a negative hyperbolic rotopulsating orbit for the equations of motion if and only if

$$\dot\beta = \frac{a}{\sum_{j=1}^n m_j(w_j^2+x_j^2+1)},$$

where a is a constant, and the variables $w_i, x_i,\ i=\overline{1,n}$, satisfy the system of $2n$ second-order differential equations

$$\begin{cases} \ddot w_i = \sum_{\substack{j=1\\ j\neq i}}^n \frac{m_j(w_j+\nu_{ij}w_i)}{(\nu_{ij}^2-1)^{3/2}} + H_i w_i \\ \ddot x_i = \sum_{\substack{j=1\\ j\neq i}}^n \frac{m_j(x_j+\nu_{ij}x_i)}{(\nu_{ij}^2-1)^{3/2}} + H_i x_i, \end{cases}$$

where

$$H_i := \frac{(w_i\dot x_i - x_i\dot w_i)^2+\dot w_i^2+\dot x_i^2}{w_i^2+x_i^2+1} + \frac{a^2(w_i^2+x_i^2+1)}{[\sum_{j=1}^n m_j(w_j^2+x_j^2+1)]^2},$$

$i=\overline{1,n}$, and, for any $i,j\in\{1,2,\dots,n\}$, ν_{ij} is given by

$$\nu_{ij} := \mathbf{q}_i \boxdot \mathbf{q_j} = w_iw_j + x_ix_j - (w_i^2+x_i^2+1)^{\frac12}(w_j^2+x_j^2+1)^{\frac12}\cosh(b_i-b_j).$$

Examples

Negative hyperbolic Eulerian RP orbits

$$m_1 = m_2 = m_3 =: m > 0, \quad w^2 + x^2 - \eta^2 = -1$$

$$\mathbf{q} = (\mathbf{q}_1, \mathbf{q}_2, \mathbf{q}_3), \quad \mathbf{q}_i = (w_i, x_i, y_i, z_i), \quad i = 1, 2, 3,$$
$$w_1 = 0, \quad x_1 = 0, \quad y_1 = \sinh\beta(t), \quad z_1 = \cosh\beta(t),$$
$$w_1 = w(t), \quad x_1 = x(t), \quad y_1 = \eta(t)\sinh\beta(t), \quad z_1 = \eta(t)\cosh\beta(t),$$
$$w_1 = -w(t), \quad x_1 = -x(t), \quad y_1 = \eta(t)\sinh\beta(t), \quad z_1 = \eta(t)\cosh\beta(t),$$

The equations of motion reduce to the system

$$\begin{cases} \dot{x} = \eta \\ \dot{\eta} = \left[\frac{h}{m} + \frac{m[4\zeta^2x^4 - 2\zeta x^2 + 1]}{(2\zeta x^2+1)^{\frac{1}{2}}(2\zeta x^2-1)^{\frac{3}{2}}} - \frac{m(\zeta x^2)^{\frac{1}{2}}(2\zeta x^2-3)}{(\zeta x^2-1)^{\frac{3}{2}}} - \frac{a^2}{2m^2(2\zeta x^2+3)^2}\right]x, \end{cases}$$

with $\zeta \geq 1$ constant, so existence and uniqueness follows.

Negative elliptic-hyperbolic RP orbits

A solution of the equations of motion in $\mathbb{H}^3$ is called a negative elliptic-hyperbolic rotopulsating orbit if it is of the form

$$\mathbf{q} = (\mathbf{q}_1, \mathbf{q}_2, \dots, \mathbf{q}_N), \quad \mathbf{q}_i = (w_i, x_i, y_i, z_i), \quad i = \overline{1, n},$$
$$w_i = r_i(t)\cos[\alpha(t) + a_i], \quad x_i = r_i(t)\sin[\alpha(t) + a_i],$$
$$y_i = \eta_i(t)\sinh[\beta(t) + b_i], \; z_i = \eta_i(t)\cosh[\beta(t) + b_i],$$

where $a_i, b_i, \; i = \overline{1, n}$, are constants, α and β are not constant functions, whereas r_i, η_i, and z_i satisfy the conditions

$$z_i \geq 1 \quad \text{and} \quad r_i^2 - \eta_i^2 = -1, \quad i = \overline{1, n}.$$

When r and η are constant and $\alpha(t) = \bar{\alpha} t, \beta(t) = \bar{\beta} t$, with $\bar{\alpha}, \bar{\beta}$ nonzero constants, then the solution is called a negative elliptic-hyperbolic relative equilibrium.

Criterion for negative elliptic-hyperbolic RP orbits

A solution candidate as above is a negative elliptic-hyperbolic rotopulsating orbit for the equations of motion if and only if

$$\dot{\alpha} = \frac{d_1}{\sum_{i=1}^{n} m_i r_i^2}, \quad \dot{\beta} = \frac{d_2}{M + \sum_{i=1}^{n} m_i r_i^2},$$

with d_1, d_2 constants, and the variables $r_i,\ i = \overline{1, n}$, satisfy the n second-order differential equations

$$\ddot{r}_i = r_i(1 + r_i^2)\left[\frac{d_1^2}{(\sum_{i=1}^{n} m_i r_i^2)^2} - \frac{d_2^2}{(M + \sum_{i=1}^{n} m_i r_i^2)^2}\right] + \frac{r_i \dot{r}_i^2}{1 + r_i^2}$$
$$+ \sum_{\substack{j=1 \\ j \neq i}}^{n} \frac{m_j[r_j(1 + r_i^2)\cos(a_i - a_j) - r_i(1 + r_i^2)^{\frac{1}{2}}(1 + r_j^2)^{\frac{1}{2}}\cosh(b_i - b_j)]}{(\delta_{ij}^2 - 1)^{\frac{3}{2}}},$$

where, for any $i, j \in \{1, 2, \ldots, n\}$ with $i \neq j$, we denoted

$$\delta_{ij} := \mathbf{q}_i \boxdot \mathbf{q}_j = r_i r_j \cos(a_i - a_j) - (1 + r_i^2)^{\frac{1}{2}}(1 + r_j^2)^{\frac{1}{2}}\cosh(b_i - b_j).$$

Examples

Negative elliptic-hyperbolic Eulerian RP orbits

$$m_1 = m_2 = m_3 := m > 0, \quad r^2 - \eta^2 = -1,$$

$$\mathbf{q} = (\mathbf{q}_1, \mathbf{q}_2, \mathbf{q}_3), \ \ \mathbf{q}_i = (w_i, x_i, y_i, z_i), \ \ i = 1, 2, 3,$$

$$w_1 = 0, \ x_1 = 0, \ y_1 = \sinh\beta(t), \ z_1(t) = \cosh\beta(t),$$

$$w_2 = r(t)\cos\alpha(t), \ x_2 = r(t)\sin\alpha(t), \ y_2 = \eta(t)\sinh\beta(t), \ z_2(t) = \eta(t)\cosh\beta(t),$$

$$w_3 = -r(t)\cos\alpha(t), \ x_3 = -r(t)\sin\alpha(t), \ y_3 = \eta(t)\sinh\beta(t), \ z_3 = \eta(t)\cosh\beta(t),$$

The equations of motion reduce to the system

$$\begin{cases} \dot{r} = \rho \\ \dot{\rho} = r(1+r^2)\left[\frac{d_1^2}{4m^2r^4} - \frac{d_2^2}{m^2(3+2r^2)^2}\right] + \frac{r\rho^2}{1+r^2} - \frac{m(5+4r^2)}{4r^2(1+r^2)^{1/2}}, \end{cases}$$

which leads to the desired existence and uniqueness results for admissible initial conditions.

Moltes gràcies!

Muchas gracias!

Merci beaucoup!

Thank you very much!

Vielen Dank!