Local integrability and linearizability of three dimensional Lotka-Volterra systems

Waleed Aziz and Colin Christopher

School of Computing and Mathematics
University of Plymouth

October 3rd, 2012,
New Trends in Dynamical Systems, Salou

Outline

- Introduction.

Outline

- Introduction.
- Basic Definitions.

Outline

- Introduction.
- Basic Definitions.
- Mechanisms of Integrability and Linearizability Conditions.

Outline

- Introduction.
- Basic Definitions.
- Mechanisms of Integrability and Linearizability Conditions.
- Some Results.

Outline

- Introduction.
- Basic Definitions.
- Mechanisms of Integrability and Linearizability Conditions.
- Some Results.
- Monodromy Arguments.

Outline

- Introduction.
- Basic Definitions.
- Mechanisms of Integrability and Linearizability Conditions.
- Some Results.
- Monodromy Arguments.
- Extension of Singer's Theorem.
- Applications to Bifurcation Theory.

Background

- Two Dimensional Systems

$$\dot{x} = \mu x + P(x, y), \quad \dot{y} = -\lambda y + Q(x, y)$$

  - (1:-1)-resonant quadratic systems (Dulac and Kapteyn)
  - (1:-1)-resonant Homogeneous Cubic systems (Sibirskii)
  - (1:-2)-resonant Center (by Fronville, Sadovski and Żołądek)

Background

- Two Dimensional Systems

$$\dot{x} = \mu x + P(x, y), \quad \dot{y} = -\lambda y + Q(x, y)$$

  - (1:-1)-resonant quadratic systems (Dulac and Kapteyn)
  - (1:-1)-resonant Homogeneous Cubic systems (Sibirskii)
  - (1:-2)-resonant Center (by Fronville, Sadovski and Żołądek)

- Three Dimensional Systems

$$\dot{x} = \lambda x + P(x, y, z), \quad \dot{y} = -\mu y + Q(x, y, z) \quad \dot{z} = \nu z + R(x, y, z)$$

  - ABC System (Moulin-Ollagnier)
  - With One Zero Eigenvalue (Basov and Romanovski)

General form of Three Dimensional Lotka-Volterra systems

Three dimensional Lotka-Volterra system has the form

$$\begin{aligned}\dot{x} &= x(\lambda + ax + by + cz) &= P,\\ \dot{y} &= y(\mu + dx + ey + fz) &= Q,\\ \dot{z} &= z(\nu + gx + hy + kz) &= R,\end{aligned}$$

where $\lambda, \nu, \mu \neq 0$.

The scientific literature on Lotka-Volterra systems is very extensive due to their many applications such as:

- Population Dynamics
- Ecology
- Chemistry
- Game Theory etc.

Some basic definitions

- First Integral: $\mathcal{X}H = P\frac{\partial H}{\partial x} + Q\frac{\partial H}{\partial y} + R\frac{\partial H}{\partial z} = 0$

Some basic definitions

- First Integral: $\mathcal{X}H = P\frac{\partial H}{\partial x} + Q\frac{\partial H}{\partial y} + R\frac{\partial H}{\partial z} = 0$
- Invariant Algebraic Surface: $\mathcal{X}F = P\frac{\partial F}{\partial x} + Q\frac{\partial F}{\partial y} + R\frac{\partial F}{\partial z} = CF$

Some basic definitions

- First Integral: $\mathcal{X}H = P\frac{\partial H}{\partial x} + Q\frac{\partial H}{\partial y} + R\frac{\partial H}{\partial z} = 0$
- Invariant Algebraic Surface: $\mathcal{X}F = P\frac{\partial F}{\partial x} + Q\frac{\partial F}{\partial y} + R\frac{\partial F}{\partial z} = CF$
- Integrable: Can be brought to the form

  $$\dot{X} = \lambda X m, \quad \dot{Y} = \mu Y m, \quad \dot{Z} = \nu Z m,$$

  after a change of variables where $m = 1 + O(X, Y, Z)$.
  Equivalently, $\exists$ ϕ and ψ first integrals, with

  $\phi = x^{-\mu} y^{\lambda}(1 + O(x, y, z))$ and $\psi = y^{\nu} z^{-\mu}(1 + O(x, y, z))$

Some basic definitions

- First Integral: $\mathcal{X}H = P\frac{\partial H}{\partial x} + Q\frac{\partial H}{\partial y} + R\frac{\partial H}{\partial z} = 0$
- Invariant Algebraic Surface: $\mathcal{X}F = P\frac{\partial F}{\partial x} + Q\frac{\partial F}{\partial y} + R\frac{\partial F}{\partial z} = CF$
- Integrable: Can be brought to the form

$$\dot{X} = \lambda X m, \quad \dot{Y} = \mu Y m, \quad \dot{Z} = \nu Z m,$$

after a change of variables where $m = 1 + O(X, Y, Z)$.
Equivalently, $\exists$ ϕ and ψ first integrals, with

$$\phi = x^{-\mu} y^{\lambda}(1 + O(x, y, z)) \quad \text{and} \quad \psi = y^{\nu} z^{-\mu}(1 + O(x, y, z))$$

- Linearizable: Can be brought to the form

$$\dot{X} = \lambda X, \quad \dot{Y} = \mu Y, \quad \dot{Z} = \nu Z,$$

after a change of variables.

Necessary and sufficient conditions for integrability

To find necessary and sufficient conditions for integrability:

Step1: We seek two analytic first integrals of the form

$$\phi = x^{-\mu} y^{\lambda} \big(1 + o(x, y, z)\big) \quad \text{and} \quad \psi = y^{\nu} z^{-\mu} \big(1 + o(x, y, z)\big)$$

where λ, ν and $\mu < 0$.

Step2: We then calculate the successive terms in the power series expansion of $\mathcal{X}\phi = 0$ and $\mathcal{X}\psi = 0$. The obstructions to the existence of ϕ and ψ correspond to the resonant terms in the normal form of the vector field.

Mechanism for integrability

Step3: Having calculated a number of these quantities, we then solve them simultaneously by computing a Gröbner basis. The conditions are necessary, but we do not know as yet that they are sufficient. The calculations were performed in MAPLE and REDUCE. Finally the minAssGTZ algorithm in SINGULAR was used to check that the conditions found were irreducible.

Step4: We need finally to prove sufficiency of these conditions by exhibiting two independent first integrals via the Darboux method together with inverse Jacobi multipliers or some other technique like blow-downs or the existence of a linearizable node.

Mechanism for Linearizability

We seek a change of coordinates

$$X = x + o(x, y, z), \quad Y = y + o(x, y, z), \quad Z = z + o(x, y, z)$$

which brings the system to

$$\dot{X} = \lambda X, \quad \dot{Y} = \mu Y, \quad \dot{Z} = \nu Z,$$

Similar to integrability mechanism, we find factorized Gröbner basis by MAPLE, REDUCE and SINGULAR and then prove their sufficiency.

Results (Inverse Jacobi Multiplier)

A function M is an inverse Jacobi multiplier for the vector field $\mathcal{X}$ if

$$\mathcal{X}(M) = M\text{div}(X) \qquad \Longleftrightarrow \qquad \text{div}(\mathcal{X}/M) = 0.$$

Suppose that the level surfaces $\phi = c$ are locally parameterized by some function $z = f_c(x, y)$. Using the x and y coordinates to parameterize $\phi = c$, we obtain a vector field

$$P(x, y, f_c(x, y))\frac{\partial}{\partial x} + Q(x, y, f_c(x, y))\frac{\partial}{\partial y}.$$

It was proven that

$$M(x, y, f_c(x, y))\frac{\partial \phi}{\partial z}(x, y, f_c(x, y))$$

is an inverse integrating factor for this vector field. Hence, by quadratures along $\phi = c$, we can construct a second first integral $\psi_c(x, y)$ for each value of c. The function $\psi_{\phi(x,y,z)}(x, y)$ gives a second first integral of the system.

Results (Inverse Jacobi Multiplier)

Theorem

Suppose the analytic vector field

$$x(\lambda+ax+by+cz)\frac{\partial}{\partial x}+y(\mu+dx+ey+fz)\frac{\partial}{\partial y}+z(\nu+gx+hy+kz)\frac{\partial}{\partial z},$$

has an analytic first integral $\phi = x^\alpha y^\beta z^\gamma(1+O(x,y,z))$ *with at least one of* α, β, $\gamma \neq 0$ *and a Jacobi multiplier* $M = x^r y^s z^t(1+O(x,y,z))$ *and suppose that the cross product of* $(r-i-1, s-j-1, t-k-1)$ *and* (α,β,γ) *is bounded away from zero for any integers* $i,j,k \geq 0$*, then the system has a second analytic first integral of the form* $\psi = x^{1-r}y^{1-s}z^{1-t}(1+O(x,y,z))$*, and hence the system* (1) *is integrable.*

Relation Between Integrability and Linearizability

Theorem

Consider the three dimensional Lotka-Volterra system

$$\begin{aligned}\dot{x} &= x(\lambda + ax + by + cz) &= P,\\ \dot{y} &= y(\mu + dx + ey + fz) &= Q,\\ \dot{z} &= z(\nu + gx + hy + kz) &= R,\end{aligned}$$

$x = 0$, $y = 0$ *and* $z = 0$ *have cofactors* L_x, L_y *and* L_z *respectively. If* L_x, L_y, L_z *and the divergence* $div(X)$ *are linearly independent then the origin is integrable if and only if it is linearizable.*

Results

We consider three dimensional Lotka-Volterra system

$$\begin{aligned} \dot{x} &= x(\lambda + ax + by + cz) &= P, \\ \dot{y} &= y(\mu + dx + ey + fz) &= Q, \\ \dot{z} &= z(\nu + gx + hy + kz) &= R. \end{aligned} \tag{1}$$

We give a complete classification of the integrability and linearizability conditions for (1) at the origin in the case of $(1:-1:1)$, $(2:-1:1)$ or $(1:-2:1)$-resonance.

(1:-1:1)-Resonance

Theorem

The origin of system (1) with $1 : -1 : 1$ resonant is integrable if and only if one of the following conditions holds:

1) $ab - de = ac - 2ak + gk = ae + ah - de - eg = af + ak - dk - gk = bd + bg - de - dh = bf - ch - fh + hk = bk - ce + ek - hk = cd + cg - 2dk + fg - gk = ef - hk = 0$

2) $b = d = f = h = 0$

3) $f = g = h = b - e = d - a = 0$

3^*) $b = c = d = f - k = e - h = 0$

4) $b = c = d = f = k = 0$

4^*) $a = d = g = h = f = 0$

5) $b = e = h = 0$

(1:-1:1)-Resonance

Theorem

Moreover, the system is linearizable if and only if either one of the conditions (2)-(5) or one of the following holds:

$$1.1)\ a = c = d = f = g = k = 0$$

$$1.2)\ a = bk - ch = d = e - h = f - k = g = 0$$

$$1.2^{*})\ a - d = b - e = c = dh - eg = f = k = 0$$

$$1.3)\ a - g = b - h = c - k = d - g = e - h = f - k = 0$$

Methods Used

To find two independent first integrals we have used the following tools:

- Darboux (Invariant algebraic surfaces and exponential factors)
- Darboux with Inverse Jacobi multiplier
- Blow-down method
- Linearizable node
- Power Series with Inverse Jacobi multiplier

(1:-1:1)-Resonance (Darboux Method)

We now prove the sufficiency of some of the conditions above:

Case 1: If $e \neq 0$, the system has an invariant algebraic surface

$$\ell = 1 + ax - ey + kz = 0$$

with cofactor

$$L_\ell = ax + ey + kz$$

Two independent first integrals are

$$\phi_1 = xy\ell^{-1-\frac{b}{e}}, \qquad \phi_2 = yz\ell^{-1-\frac{h}{e}}.$$

Application to (1:-1:1) Resonant (Darboux Method)

When $e = 0$, we have some sub cases:

for example when $b, h \neq 0$. We get an exponential factor

$$\ell = \exp(dhx - bhy + bfz)$$

with cofactor

$$dhx + bhy + bfz.$$

This gives first integrals

$$\phi_1 = xy\ell^{-\frac{1}{h}}$$

and

$$\phi_2 = yz\ell^{-\frac{1}{b}}.$$

(1:-1:1)-Resonance (Darboux Linearization)

Case 1.1: The system appears as

$$\dot{x} = x(1 + by), \quad \dot{y} = y(-1 + ey), \quad \dot{z} = z(1 + hy),$$

If $e \neq 0$, the change of coordinates

$$(X, Y, Z) = (x(1 - ey)^{-\frac{b}{e}}, y(1 - ey)^{-1}, z(1 - ey)^{-\frac{h}{e}})$$

linearizes the system.

(1:-1:1)-Resonance (Linearizable Node)

Case 5: Then the system reduces to

$$\dot{x} = x(1 + ax + cz), \quad \dot{y} = y(-1 + dx + fz), \quad \dot{z} = z(1 + gx + kz),$$

The first and third equations give a linearizable node and hence we transform to

$$\dot{X} = X, \qquad \dot{Z} = Z.$$

Thus $\frac{\dot{y}}{y} = (-1 + dx(X, Z) + fz(X, Z))$. It is suffices to find a function $\ell(X, Z)$ such that $\dot{\ell}(X, Z) = dx(X, Z) + fz(X, Z)$, then the transformation $Y = ye^{-\ell}$ gives $\dot{Y} = -Y$.

Writing $\ell(X, Z) = \sum_{i+j>0} b_{ij} X^i Z^j$, we have to solve $\dot{\ell} = \sum_{i+j>0} (i + j) b_{ij} X^i Z^j = dx(X, Z) + fz(X, Z) = \sum_{i+j>0} a_{ij} X^i Z^j$, then $b_{ij} = \frac{a_{ij}}{i+j}$, so it is clear the solution exists and is analytic.

(2:-1:1) and (1:-2:1)-Resonance

For (2:-1:1)-resonance, we have 11 cases for integrability with 13 cases of linearizability. While for (1:-2:1)-resonance, we have 21 integrability with 19 linearizability conditions.

We have selected different cases from both (2:-1:1) and (1:-2:1)-resonant.

(2:-1:1)-Resonance (Blow-down Method)

Case 3: The system is

$$\dot{x} = x(2+ax-ey-fz),\ \dot{y} = y(-1+ey+fz),\ \dot{z} = z(1+gx+ey+fz). \tag{2}$$

Using $(X, Y, Z) = (x, xy, xz)$, the system above becomes

$$\dot{X} = 2X + aX^2 - eY - fZ,\ \dot{Y} = Y(1+aX),\ \dot{Z} = Z(3+(a+g)X).$$

The origin is in the Poincaré domain. Hence it is linearizable via

$$(\tilde{X}, \tilde{Y}, \tilde{Z}) = (X - eY + fZ + O(2), Y(1+O(1)), Z(1+O(1))).$$

The two first integrals $\tilde{\phi} = \tilde{X}^{-1}\tilde{Y}^2$ and $\tilde{\psi} = \tilde{X}^{-2}\tilde{Y}\tilde{Z}$ of the linearized system pull back to first integrals of (2) in the form

$$\phi_1 = x\,y^2(1+O(x,y,z)), \quad \text{and} \quad \phi_2 = yz(1+O(x,y,z)).$$

(2:-1:1)-Resonance (Inverse Jacobi Multiplier)

Case 7: In this case the system (1) reduces to

$\dot{x} = x(2 + ax)$, $\dot{y} = y(-1 + dx + ey + fz)$, $\dot{z} = z(1 + gx + ey + fz)$.

The system has an IAS $\ell = 2 + ax$ with cofactor $L_\ell = ax$ yielding a first integral $\phi = x^{-1} y^{-1} z \ell^{\frac{d-g+a}{a}}$. We also have an inverse Jacobi multiplier

$$IJM = x^{\frac{5}{2}} y^3 (2 + ax)^{-\frac{1}{2} - \frac{2d}{a} + \frac{g}{a}}$$

Theorem 1 therefore guarantees the existence of a second first integral of the form $\psi = x^{-3/2} y^{-2} z(1 + O(x, y, z))$. Now the desired first integrals are $\phi_1 = \phi^2 \psi^{-2} = xy^2(1 + \dots)$ and $\phi_2 = \phi^3 \psi^{-2} = yz(1 + \dots)$.

(2:-1:1)-Resonance (Linearizable Node)

Case 9: The system (1) can be written as

$$\dot{x} = x(2 + ax + cz), \quad \dot{y} = y(-1 + dx + ey), \quad \dot{z} = z(1 + gx + kz).$$

The first and third equations give a linearizable node. To linearize the second equation, we seek an invariant surface of the form $\ell + \chi y = 0$ with cofactor $dx + ey$ where $\ell = \ell(X, Z)$, $\chi = \chi(X, Z)$.

Use $Y = \frac{y}{\ell + \chi y}$ to linearize the second equation. To find such ℓ and χ we therefore need to solve

(2:-1:1)-Resonance (Linearizable Node)

$$\dot{\chi} y - \chi y = \ell(dx + ey) - \dot{\ell} \Rightarrow \dot{\chi} - \chi = e\,\ell, \quad \dot{\ell} = d\,x\,\ell.$$

To find ℓ, we write $\ell = e^{\psi}$ and solve $\dot{\psi} = dx$.

Let $\psi = \sum_{i+j>0} c_{ij} X^i Z^j$, then

$$\sum_{i+j>0} (2i + j) c_{ij} X^i Z^j = dx(X, Z) = d\,X + \sum_{i+j>1} d_{ij} X^i Z^j,$$

for some d_{ij}.

(2:-1:1)-Resonance (Linearizable Node)

Clearly, $c_{10} = \frac{d}{2}$, $c_{01} = 0$, $c_{ij} = \frac{d_{ij}}{2i+j}$ for $i + j > 1$. The convergence of $\sum_{i+j>1} d_{ij} X^i Z^j$, guarantees the convergence of ψ and hence ℓ. Furthermore, it is clear that ℓ will contain no term in Z. Now, writing $\ell = \sum b_{ij} X^i Z^j$, and noting that $a_{01} = 0$, we find that $\chi = \sum \frac{e}{2i+j-1} a_{ij} X^i Z^j$ gives a convergent expression for χ.

(1:-2:1)-Resonance (Projective Transformation)

Case 3: The reduced system is therefore

$$\dot{x} = x(1+2gx-ey-3fz), \quad \dot{y} = y(-2+ey+fz), \quad \dot{z} = z(1+gx-fz).$$

When $fg \neq 0$, apply a transformation of the form $(X, Y, Z) = (gx - fz, xy, z^2)$, the resulting system is

$$\dot{X} = X + 2X^2 - f^2Z - geY, \quad \dot{Y} = Y(-1+2X), \quad \dot{Z} = Z(2+2X),$$

Finally we apply the projective transformation

$$(\hat{X}, \hat{Y}, \hat{Z}) = (\frac{X}{Y}, \frac{1}{Y}, \frac{Z}{Y})$$

(1:-2:1)-Resonance (Projective Transformation)

to get the linear system

$$\dot{\hat{X}} = 2\hat{X} - f^2\hat{Z} - ge, \quad \dot{\hat{Y}} = \hat{Y} - 2\hat{X}, \quad \dot{\hat{Z}} = 3\hat{Z}.$$

This system admit first integrals $\phi = \ell_1^{-2}\ell_2, \qquad \psi = Z\,\ell_1^{-3},$ where

$$\ell_1 = 1 - \frac{1}{ge}\hat{X} - \frac{1}{2ge}\hat{Y} - \frac{f^2}{2ge}\hat{Z}, \quad \ell_2 = 1 - \frac{2}{ge}\hat{X} - \frac{2f^2}{ge}\hat{Z},$$

One can find two independent first integrals of the desirable form

$$\phi_1 = \frac{\phi}{2f} - \sqrt{\psi} = x^2\,y(-\frac{g}{f} + \dots)$$

and

$$\phi_2 = \frac{\psi}{\phi_1} = y\,z^2(\frac{f}{g} + \dots).$$

(1:-2:1)-Resonance (Transforming to Linearizable Node)

Case 10: In this case we have the system

$$\dot{x} = x(1 + ax), \quad \dot{y} = y(-2 + ey + fz), \quad \dot{z} = z(1 + gx + ey + fz),$$

Use $(Y, Z) = \left(\frac{y}{2+2fz-ey}, \frac{z}{2+2fz-ey}\right)$ to gives a new system

$$\dot{x} = x(1 + ax), \quad \dot{Y} = -2Y(1 + fgxZ), \quad \dot{Z} = Z(1 + gx - 2fgxZ),$$

The first and the third equations obviously gives a linearizable node. To linearize the second equation, it is suffices to find $\psi(\hat{X}, \hat{Z})$ such that $\dot{\psi} = fgxZ$ and use $\hat{Y} = Ye^{2\psi}$.

Conclusion

Table: Classification of all cases

Methods	(1:-1:1)	(2:-1:1)	(1:-2:1)
Darboux Method	6	8	21
Darboux with IJM	0	1	5
Linearizable Node	5	4	4
Blow-down Method	0	1	0
Power Series with IJM	0	1	0

Generalization of the Lotka-Volterra systems

The calculations have been extended to the case of (3,-1,2)-Resonance and also systems of the form

$$\begin{aligned}
\dot{x} &= x(\lambda + ax + by + cz) &&= P,\\
\dot{y} &= \mu y + dx^2 + exy + fxz + gyz + hy^2 + kz^2 &&= Q,\\
\dot{z} &= z(\nu + gx + hy + kz) &&= R,
\end{aligned}$$

These case are more challenging computationally than the Lotka-Volterra systems.

Case 3 (Ricatti Equation)

The system has the form

$$\dot{x} = x(1 + by), \quad \dot{y} = -y + fxz + hy^2, \quad \dot{z} = z(1 - by).$$

The invariant algebraic surface is $F = 1 - 2hy + fhxz + h^2y^2 = 0$ with cofactor $C_F = 2hy$. Then $(X, Z) = (x\,F^{-\frac{b}{2h}}, z\,F^{\frac{b}{2h}})$ linearizes the first and third equations. Second equation is a Riccati equation. We seek a solution the form $y = G(xz)$, then

$$G'(xz) = \frac{f}{2} - \frac{1}{2xz}G(xz) + \frac{h}{2xz}G^2(xz)$$

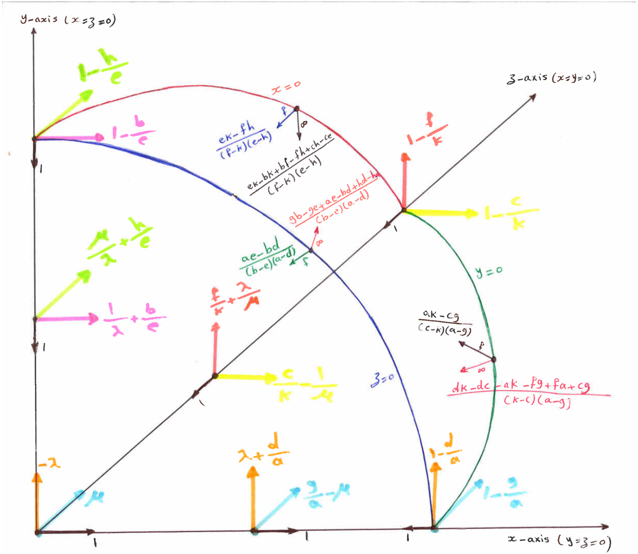
which has a particular solution $y_1 = \frac{sin(\sqrt{fhxz}) - cos(\sqrt{fhxz})\sqrt{fhxz})}{hsin(\sqrt{fhxz})}$. The change of variables $y = Y + y_1$ transform the second equation to $\dot{Y} = Y(-1 + 2hy_1 + hY)$. Look for an invariant algebraic surface of the form $\alpha(X, Z) + \beta(X, Z)Y = 0$ with cofactor $2hy_1 + hY$ and $\alpha(0, 0) = 1$, so that the transformation $\frac{Y}{\alpha + \beta Y}$ will linearize this equation.

The Monodromy Argument

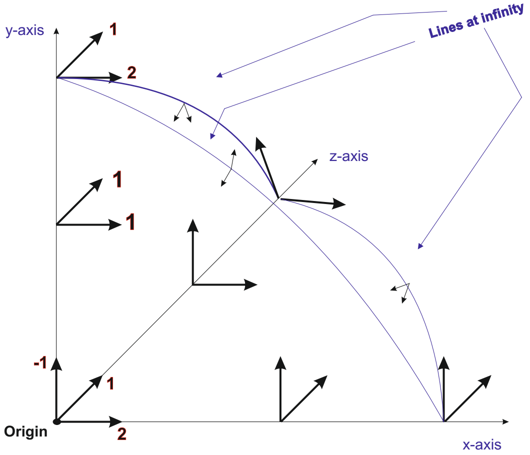
We look at the monodromy group of one of the separatrices $x = 0$, $y = 0$, $z = 0$ together with the monodromy ofthe line at infinity.

Each of these lines can be considered as a copy of Riemann sphere with three singular points on it. If one of these has trivial monodromy and the other is linearizable, then the third singular point is integrable.

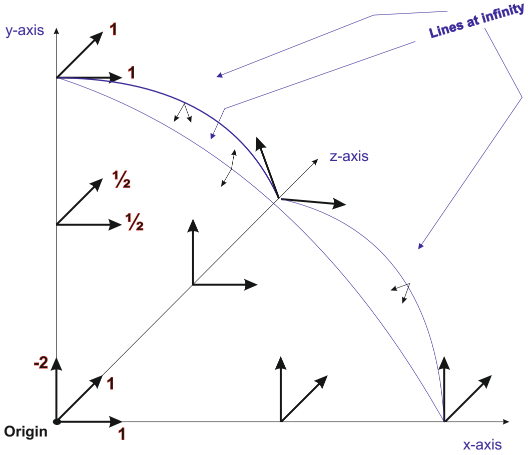
The Monodromy Argument



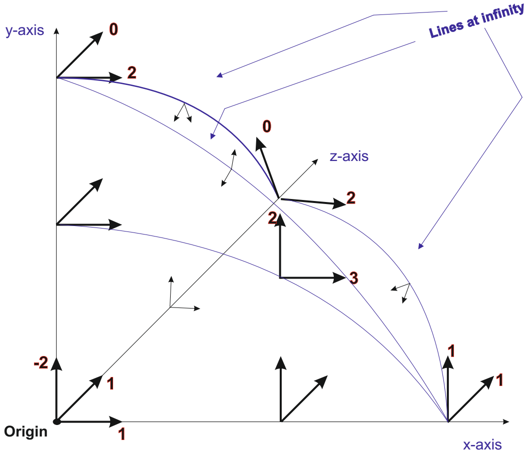
The Monodromy Argument



The Monodromy Argument



The Monodromy Argument



Liouvillian Integrability

Theorem (Extension of Singer's Theorem)

Let us consider a rational 1-form ω in $\mathbb{C}^n$. Then ω admits a Liouvillian first integral if and only if there exists a rational closed 1-form α such that $d\omega = \alpha \wedge \omega$

Definition

We say that a function is Liouvillian if it can be obtained by a sequence of extensions from rational functions:

$$\mathbb{C}(x, y, z) = K_0 \subset K_1 \cdots \subset K_n,$$

such that for each i, either

i) K_{i+1} is algebraic over K_i;

ii) $K_{i+1} = K_i(t)$ with $dt = t\delta$, $d\delta = 0$;

iii) $K_{i+1} = K_i(t)$ with $dt = \delta$, $d\delta = 0$;

Extention of Liouvillian Theorem

Definition

Writing our vector field as a 2-form
$\Omega = P\,dy\,dz + Q\,dz\,dx + R\,dx\,dy$. We say that a three dimensional vector field is Liouvillian integrable if there exists Liouvillian 1-forms ω, α and β such that

$$\omega \wedge \Omega = 0, \quad d\omega = \alpha \wedge \omega, \quad d\alpha = 0$$

($\int \exp(\int \alpha)$ is a first integral), and

$$d\Omega = \beta \wedge \Omega, \quad d\beta = 0.$$

($\int \exp(\beta)$ is an inverse Jacobi Multiplier).

Can we choose α, β and ω to be rational 1-forms?

Generating Limit Cycles from Centers

One application of finding interesting centers in the planar case is to obtain good estimates of the number of limit cycles which can bifurcate from the center under perturbation. Does the same hold in three dimensions?

Questions

Thank you for listening