Global phase portraits of cubic systems having a center simultaneously at the origin and at infinity

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Abstract

The center problem is one of the celebrated problems in the qualitative theory of planar differential equations which is closely related to bifurcation problems of limit cycles. Only quadratic centers are completely known. For cubic centers merely partial results are known. In [1,2,3,4] the 8-parameter family of cubic differential equations (1) is considered and a complete characterization is obtained for (1) to have simultaneously a center at the origin and at infinity. Independently in [1] and [2] two normal forms are obtained representing a Hamiltonian and a reversible class. Both methods are based on the computation of Lyapunov quantities though the method in [2] has a smaller cost of computation time and generates a different pair of normal forms. Next in [2,3,4] stress is put on a systematic study of the global phase portraits of (1) aiming at a better understanding of the center mechanism in polynomial differential equations of degree 3 or higher and at providing a framework from which bifurcations of limit cycles can be studied. In this poster we recall the main results from [2,3,4] on topological classification of the global phase portraits of (1) and illustrate some of the key ingredients for their proofs.

1. Introduction

Theorem 1. *[1,2] The family of cubic differential equations*

$$\begin{aligned}\dot{x} &= \delta x - y + ax^2 + bxy + cy^2 + (dx - y)(x^2 + y^2),\\ \dot{y} &= x + \delta y + ex^2 + fxy + gy^2 + (x + dy)(x^2 + y^2),\end{aligned} \qquad (1)$$

has a center at the origin and at infinity if and only if $\delta = d = 0$ and it is Hamiltonian or reversible. That is, after rotation, system (1) can be written as (4) for the Hamiltonian class or as (5) for the reversible class (with respect to the change $(x, y, t) \to (x, -y, -t)$).

In polar coordinates (r, θ) the differential system (1) reads as

$$\begin{aligned}\dot{r} &= r^2 A(\theta),\\ \dot{\theta} &= 1 + rB(\theta) + r^2,\end{aligned} \qquad (2)$$

for some cubic homogeneous trigonometric polynomials A, B with $A(\theta + \pi) = -A(\theta), B(\theta + \pi) = -B(\theta)$.

Nontrivial singularities $(x^*, y^*) = (r^* \cos\theta^*, r^* \sin\theta^*)$ exist only for $B(\theta^*) \le -2$ and satisfy $A(\theta^*) = 0$ and

$$r^*_\pm = \frac{-B(\theta^*) \pm \sqrt{(B(\theta^*))^2 - 4}}{2}. \qquad (3)$$

2. Hamiltonian class

Theorem 2. *[2] Up to topological equivalence there are 22 different global phase portraits for the Hamiltonian vector fields of (1), i.e.*

$$\begin{aligned}\dot{x} &= -y - 2gxy + cy^2 - y(x^2 + y^2),\\ \dot{y} &= x + ex^2 + gy^2 + x(x^2 + y^2),\end{aligned} \qquad (4)$$

which can be classified in terms of the number of its singularities $\#_s$ and the relative Hamiltonian values at their singularities. Let $m(A)$ be the maximal multiplicity of A on $[0, \pi)$. If there exists $0 = \theta_0 < \theta_1 < \theta_2 < \pi$ such that $A(\theta_1) = A(\theta_2) = 0$, then let $h^i_\pm$ be the value of the Hamiltonian at $(r_\pm \cos\theta_i, r_\pm \sin\theta_i)$. For $2 \le \#_s \le 6$ the classification can be tabled as below; see [2] for $\#_s = 7$.

$\#_s$	Hamiltonian values	Condition	Figure
2	$h^0 = 1/12$		(II)
3	$h^0 = h^1 = 1/12$		(IIIa)
3	$h^0_- < 1/12$		(IIIb)
4	$h^0_- < h^1 = 1/12$		(IVa)
4	$h^0 = h^1 = h^2 = 1/12$		(IVb)
5	$h^0_- < h^1_+ < h^1_- < 1/12$	$m(A) = 2$	(Va)
5	$h^0_- = h^1_+ < h^1_- < 1/12$	$m(A) = 2$	(Vb)
5	$h^1_+ < h^0_- < h^1_- < 1/12$	$m(A) = 2$	(Vc)
5	$h^0_- < h^1_- < 1/12$	$m(A) = 1$	(Vd)
5	$h^0_+ = h^1_+ < h^0_- = h^1_- < 1/12$		(Ve)
5	$h^0_- < h^1 = h^2 = 1/12$		(Vf)
6	$h^0_- < h^1_- < h^2 = 1/12$		(VIa)
6	$h^0_+ = h^1_+ < h^0_- = h^1_- < h^2 = 1/12$		(VIb)

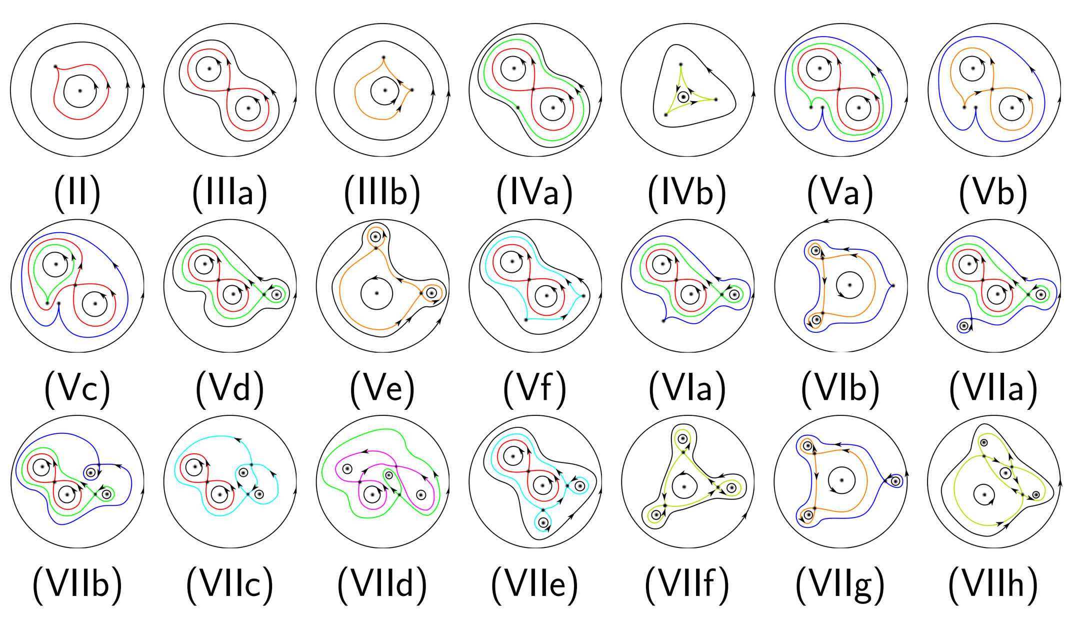
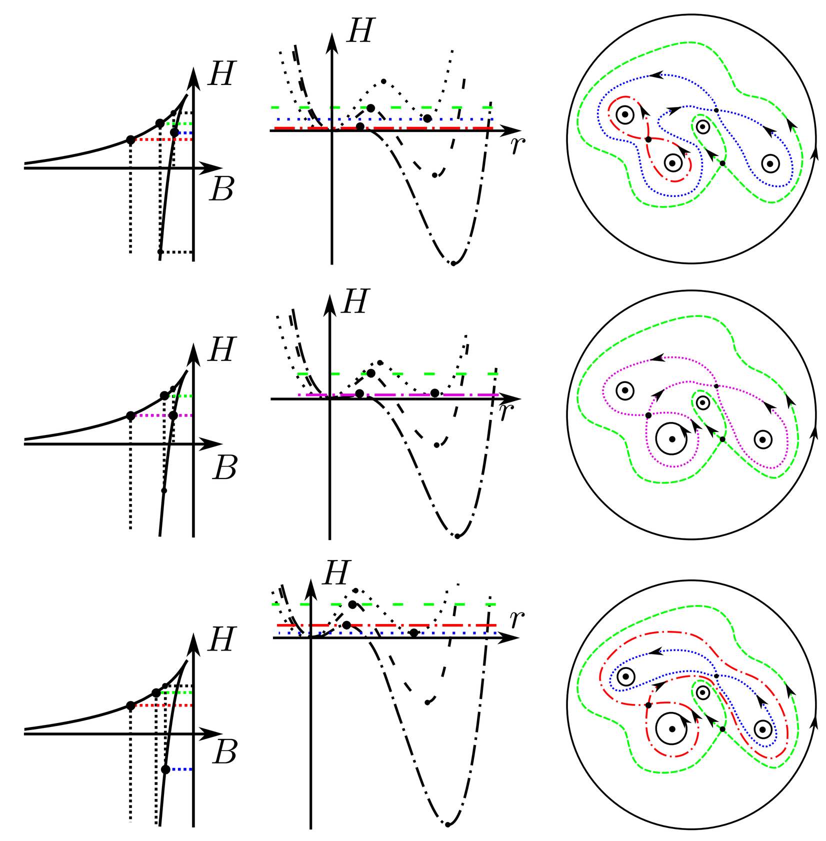


Illustration of the graphical analysis of crossing of connections:



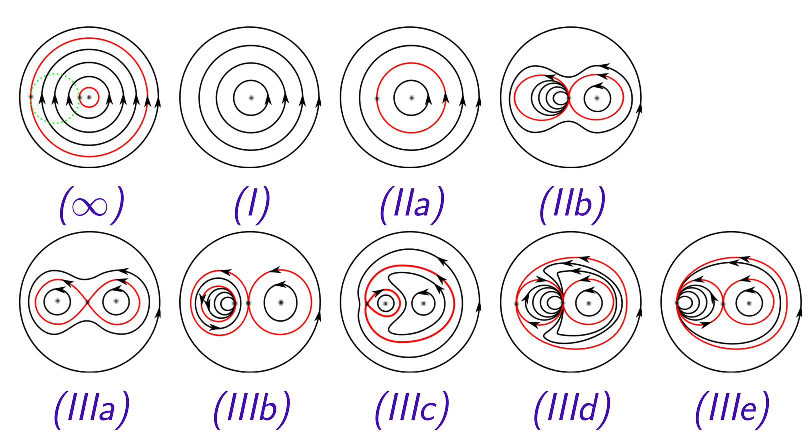
3. Reversible class with infinitely many or collinear singularities

Theorem 3. *[3] Up to topological equivalence the global phase portraits of the reversible vector fields of (1), i.e.*

$$\begin{aligned}\dot{x} &= -y + (a - 2b)xy - y(x^2 + y^2),\\ \dot{y} &= x + cx^2 + by^2 + x(x^2 + y^2),\end{aligned} \qquad (5)$$

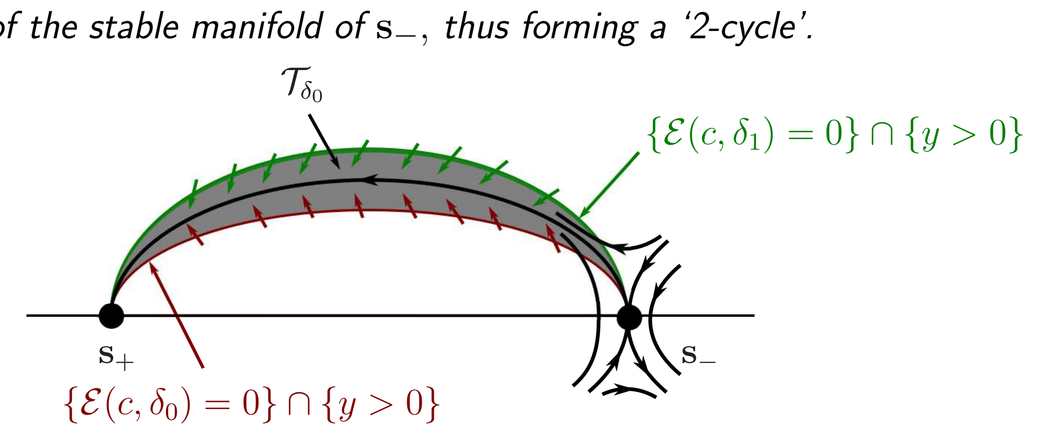
having collinear or infinitely many singularities can be classified by the nine phase portraits drawn below. For $c \ge 0$ this classification is summarized in the following table:

(a, b, c)	Figure
$(2b - a - c)\,b < 0$ $b = 0$ and $a \neq -c$ $(a - 2b + c)\,b < 0$ and $\frac{(2b-a)^2 b}{(3b-a-c)} < 4$	(I) if $0 \le c < 2$ (IIa) if $c = 2$ (IIIa) if $c > 2$ and $2b - a - c < 0$ (IIIc) if $c > 2$ and $2b - a - c > 0$
$b = a + c = 0$	(I) if $0 \le c < 2$ (IIa) if $c = 2$ (∞) if $c > 2$
$a = 2b - c$ and $b \neq 0$	(I) if $0 \le c < 2$ (IIa) if $c = 2$ and $b \le 1$ (IIb) if $c = 2$ and $b > 1$ (IIId) if $c > 2$ and $b < 0$ (IIIe) if $c > 2$ and $0 < b \le b_-(c)$ (IIIa) if $c > 2$ and $b_-(c) < b < b_+(c)$ (IIIb) if $c > 2$ and $b \ge b_+(c)$



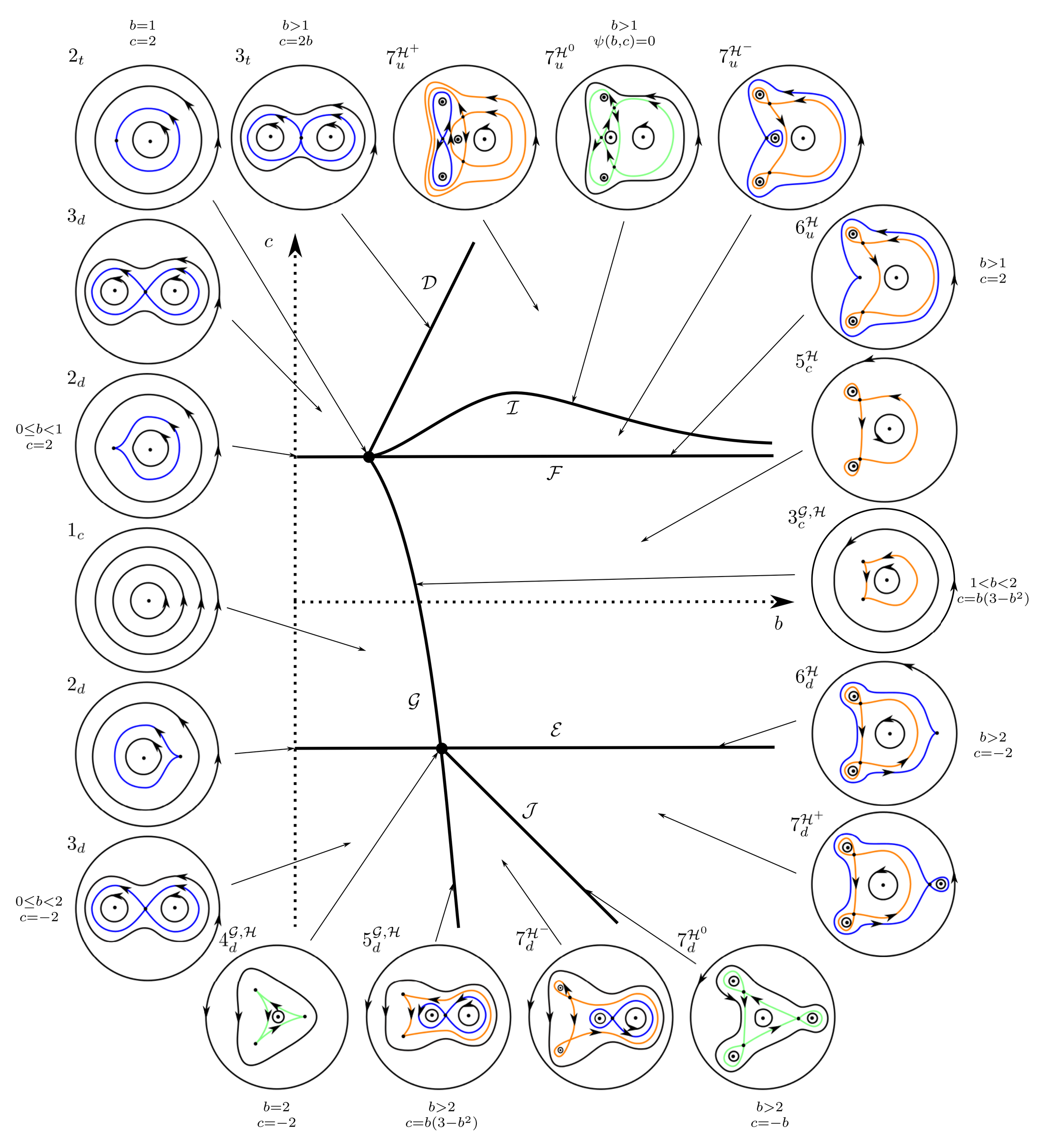
For $c < 0$ the phase portrait of $X_{(a,b,c)}$ is linearly equivalent to the one of $X_{(-a,-b,-c)}$.

Proposition. *(IIIe) Let $c > 2$, $0 < b \le b_-(c)$ and $a = c - 2b$. Then there is exactly one trajectory in the stable manifold of s_+ that forms part of the unstable manifold of s_-, see figure below; analogously there is exactly one trajectory in the unstable manifold of s_+ that forms part of the stable manifold of s_-, thus forming a '2-cycle'.*



4. Reversible Hamiltonian class

Theorem 4. *[3] The subfamily of reversible Hamiltonian differential equations of (1) having simultaneously a center at the origin and at infinity are given by (5) for $a = 0$ and the bifurcation diagram of their global phase portraits is given in next figure:*



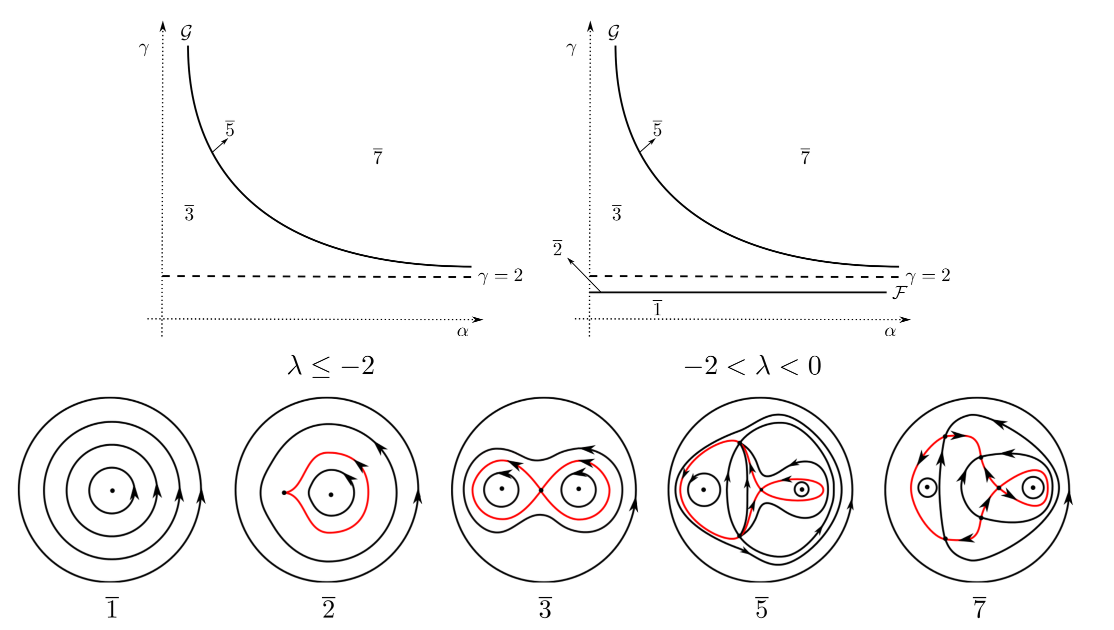
5. Reversible class with non-collinear singularities

Lemma. *The reversible cubic differential equation (5) with non-collinear singularities can be written as*

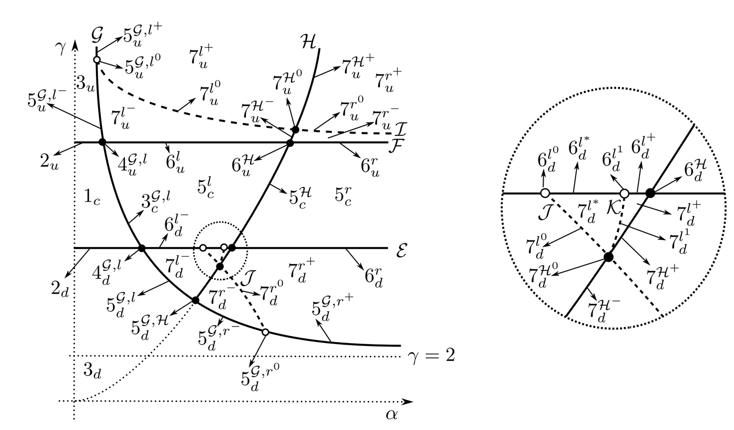
$$Y_{(\alpha,\gamma,\lambda)} \leftrightarrow \begin{cases}\dot{x} = -y - \gamma xy - y\left(x^2 + y^2\right),\\ \dot{y} = x + (\gamma - \lambda)\,x^2 + \alpha^2 \lambda y^2 + x\left(x^2 + y^2\right),\end{cases} \qquad (6)$$

where $(\alpha^2, \gamma, \lambda) = \left(\dfrac{-b}{a - 2b + c}, -a + 2b, -a + 2b - c\right)$.

Theorem 5. *[4] Up to topological equivalence the global phase portraits of (6) for $\lambda < 0$ can be classified by five phase portraits. Characteristic slices of the global bifurcation diagram for fixed $\lambda < 0$ are drawn below.*



Theorem 6. *[4] There are seven characteristic slices of the global bifurcation diagram of the global phase portraits for $\lambda = \lambda^* > 0$. The characteristic slice for fixed $\lambda^* > 6$ is drawn below; the others are similar (see [4]).*



References

[1] T. Blows, C. Rousseau, *Bifurcation at infinity of polynomial vector fields.* J. Differential Equations, 104 (1993), 215–242.

[2] M. Caubergh, J. Llibre, J. Torregrosa, *Global classification of a class of cubic vector fields whose canonical regions are period annuli.* Internat. J. Bifur. Chaos Appl. Sci. Engrg., 21(7)(2011), 1831–1867.

[3] M. Caubergh, J. Llibre, J. Torregrosa, *Global phase portraits of some reversible cubic centers with collinear or infinitely many singularities.* To appear in Internat. J. Bifur. Chaos Appl. Sci. Engrg. (2012).

[4] M. Caubergh, J. Torregrosa, *Global phase portraits of some reversible cubic centers having non-collinear singularities.* Preprint 2012.