# Nonsmooth Vector Fields on $\mathbb{R}^3$ The Cusp-Fold Singularity


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## Introduction

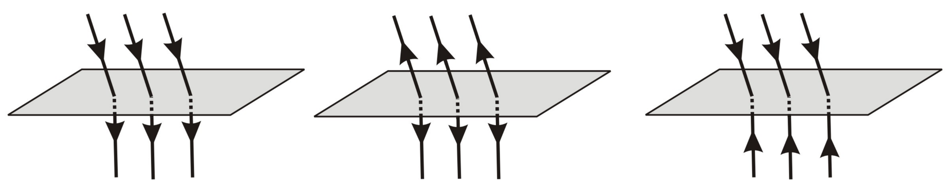
The specific topic addressed in this poster is the characterization of the *Cusp-Fold bifurcation diagram* for a specific 1-parameter family $Z_\beta$ of NSDS on $\mathbb{R}^3$ such that $Z_0$ presents a standard normal form of a Fold-Cusp singularity. In our main results the structural stability and the asymptotic stability of this singularity are discussed.

Let $K = \{(x,y,z) \in \mathbb{R}^3 \,|\, x^2+y^2+z^2 < \delta\}$, where $\delta > 0$ is arbitrarily small. Consider $\Sigma = \{(x,y,z) \in K \,|\, z = 0\}$. Clearly the *switching manifold* $\Sigma$ is the separating boundary of the regions $\Sigma_+ = \{(x,y,z) \in K \,|\, z \geq 0\}$ and $\Sigma_- = \{(x,y,z) \in K \,|\, z \leq 0\}$. Designate by $\chi^r$ the space of $C^r$-vector fields on $K$. Call $\Omega^r = \Omega^r(K,f)$ the space of vector fields $Z : K \to \mathbb{R}^3$ such that

$$Z(x,y,z) = \begin{cases} X(x,y,z), & \text{for} \quad (x,y,z) \in \Sigma_+, \\ Y(x,y,z), & \text{for} \quad (x,y,z) \in \Sigma_-, \end{cases} \tag{1}$$

where $X = (X_1, X_2, X_3)$ and $Y = (Y_1, Y_2, Y_3)$ are in $\chi^r$. We denote any element in $\Omega^r$ by $Z = (X,Y)$.
Consider the Lie derivative $X.f(p) = \langle \nabla f(p), X(p) \rangle$ and $X^i.f(p) = \langle X^{i-1}.f(p), X(p) \rangle$, $i \geq 2$ where $\langle .,. \rangle$ is the usual inner product in $\mathbb{R}^3$. We distinguish the following regions of $\Sigma$:

- **Crossing Region:** $\Sigma^c = \{p \in \Sigma | X.f)(p).(Y.f)(p) > 0\}$.
- **Sliding Region:** $\Sigma^s = \{p \in \Sigma \,|\, (X.f)(p) < 0, (Y.f)(p) > 0\}$.
- **Escaping Region:** $\Sigma^e = \{p \in \Sigma \,|\, (X.f)(p) > 0, (Y.f)(p) < 0\}$.

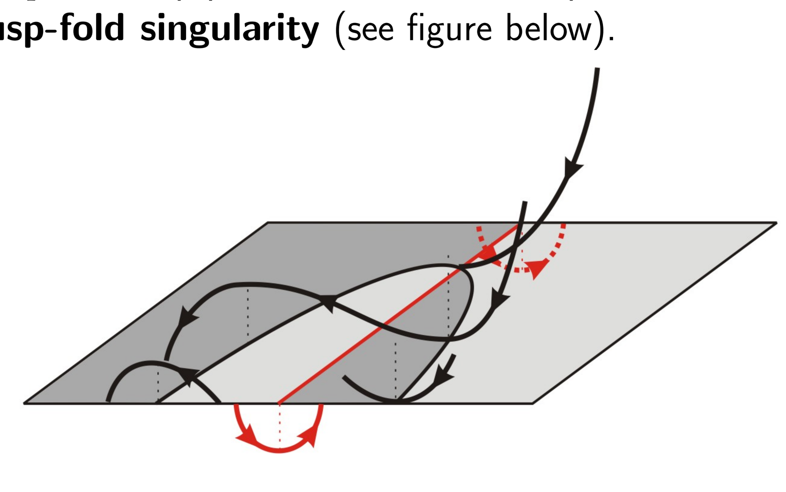


The **sliding vector field** associated to $Z \in \Omega^r$ is the vector field $Z^s$ tangent to $\Sigma^s$ and defined at $q \in \Sigma^s$ by $Z^s(q) = m - q$ with $m$ being the point of the segment joining $q + X(q)$ and $q + Y(q)$ such that $m - q$ is tangent to $\Sigma^s$ (see [5]).

We say that $q \in \Sigma$ is a $\Sigma$-**regular point** of $Z = (X,Y) \in \Omega^r$ if either $(X.f)(q).(Y.f)(q) > 0$ or $(X.f)(q).(Y.f)(q) < 0$ and $\widehat{Z}^s(q) \neq 0$ (i.e., $q \in \Sigma^s \cup \Sigma^e$ and $X(q) \nparallel Y(q)$). The points of $\Sigma$ which are not $\Sigma$-regular are called $\mathbf{\Sigma}$**-singular**. We distinguish two subsets in the set of $\Sigma$-singular points: $\Sigma^p$ and $\Sigma^t$, where $\Sigma^p = \{q \in \Sigma^e \cup \Sigma^s \,|\, \widehat{Z}^\Sigma(q) = 0\}$ is the set of **pseudo equilibria of Z** and $\Sigma^t = \{w \in \Sigma \,|\, (X.f(w))(Y.f(w)) = 0\}$ is the set of **tangential singularities of Z** (i.e., the trajectory through $w$ is tangent to $\Sigma$).

## Setting the problem

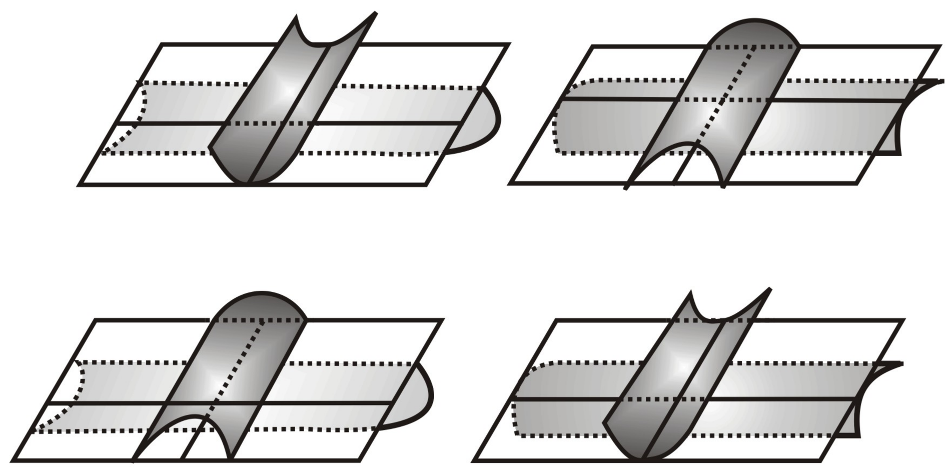
The main tool treated here concerns the contact between a general smooth vector field and the boundary $\Sigma$ of a manifold (see [2,3] for a planar analysis). In the 3-dimensional case, there are two important distinguished generic singularities: the points where this contact is either quadratic or cubic, which are called **fold points** and **cusp points** respectively. As we know by the singularity mapping theory, generically, a cusp point is an isolated point of $\Sigma$ and there are two branches of fold points emanating from it. Moreover, it is possible for a point $p \in \Sigma$ be a tangency point for both $X$ and $Y$. When $p$ is a fold point of both $X$ and $Y$ we say that $p$ is a **two-fold singularity** and when $p$ is a cusp point for $X$ and a fold point for $Y$ we say that $p$ is a **cusp-fold singularity** (see figure below).



The following 1-parameter family of piecewise smooth vector fields presents a cusp-fold singularity when $\beta = 0$:

$$Z_\beta(x,y,z) = \begin{cases} X_\beta^{b,\alpha} = \begin{pmatrix} b \\ \beta + \alpha x \\ y \end{pmatrix} & \text{if } z \geq 0, \\ Y^{a,c} = \begin{pmatrix} a \\ c \\ x \end{pmatrix} & \text{if } z \leq 0, \end{cases} \tag{2}$$

where $a\,b\,c\,\alpha\,(b+c) \neq 0$ and $\beta \in \mathbb{R}$ is an arbitrarily small parameter. Note the occurrence of all kinds of two-fold singularities when $\beta \neq 0$.



In recent years much effort has been made (see e.g. [1,4]) in order to describe the dynamics of a piecewise smooth vector field defined in a neighborhood of a two-fold singularity. This singularity is particularly relevant because in its neighborhood some of the key features of a piecewise smooth system are present: orbits that cross $\Sigma$, those that slide along it according to Filippov's convention, among others.

Let $\Gamma_+ = \{X|_{\Sigma_+}$ with $X \in \chi^r\}$ (respectively, $\Gamma_- = \{X|_{\Sigma_-}$ with $X \in \chi^r\}$). This means that $\Gamma_+$ (respectively, $\Gamma_-$) is identified with $\chi^r$. Let $\Gamma^C_{\Sigma_+} \subset \Gamma_+$ be the set of all elements $X \in \Gamma_+$ having a cusp point. $\Gamma^C_{\Sigma_+}$ is an open set in $\Gamma_+$ (see [6]). Analogously, let $\Gamma^F_{\Sigma_-} \subset \Gamma_-$ be the set of all elements $Y \in \Gamma_-$ having a fold point. $\Gamma^F_{\Sigma_-}$ is an open set in $\Gamma_+$ (see [6,8]). We denote the set of all $Z = (X,Y)$ such that $X \in \Gamma^C_{\Sigma^+}$ and $Y \in \Gamma^F_{\Sigma^-}$ by $\Gamma^{C-F}$.
Let $\Upsilon^r = \{Z^s : \Sigma^s \to T_p\Sigma \,|\, Z \in \Omega^r$ and $p \in \Sigma\}$. It is known that there exists a codimension zero submanifold $\Lambda_0^p$ of $\Upsilon^r$ (see [7]). Moreover, $\Lambda_1^p = \{\widehat{Z}^s \in \Upsilon^r \backslash \Lambda_0^p$ such that $p$ is a codimension one singularity of $\widehat{Z}^s$ and the other singularities of $\widehat{Z}^s$ has codimension zero$\}$ is a codimension one submanifold of $\Upsilon^r$ and $\Lambda_1^p$ is an open set in $\Upsilon^r \backslash \Lambda_0^p$.

Since in $\Gamma^{C-F}$ may exists vector fields whose behavior is very complicated or even chaotic, in order to restrict our analysis to a manageable set, we will deal with elements into the set $\widehat{\Omega}_1 \subset \Omega^r$ where $Z = (X,Y) \in \widehat{\Omega}_1$ if the following conditions are satisfied: (a) $Z \in \Gamma^{C-F}$; (b) $Z^s \in \Lambda_1^p$; (c) the cusp-fold singularity of $Z$ is a **Q3-singularity**; where (see [7]) a singularity $q \in \Sigma$ of the planar vector field $Z^s$ is a $Q3$-singularity if it satisfies: (a) $q$ is a cusp point of $X$ and a fold point of $Y$; (b) $X.(Y.f)(q) \neq 0$, $Y.(X.f)(q) \neq 0$ and $X.(Y.f)(q) + Y.(X.f)(q) \neq 0$; (c) $S_X \pitchfork S_Y = \{q\}$, where $S_X = \{(x,y,z) \in \Sigma \,|\, X.f(x,y,z) = X_3 = 0\}$ (respectively, $S_Y = \{(x,y,z) \in \Sigma \,|\, Y.f(x,y,z) = Y_3 = 0\}$) is the set of tangential singularities of $X$ (respectively, $Y$).

The **topological type** of $Z$ at $p \in \Sigma$ is characterized by all oriented orbits passing through or tending to $p$ (in positive or negative time).

**Definition 1.** *We say that $Z = (X,Y)$, $\widetilde{Z} = (\widetilde{X}, \widetilde{Y}) \in \Omega^r(K,f)$ presenting switching manifolds $\Sigma$ and $\widetilde{\Sigma}$, respectively, are* **mild equivalent** *if the following conditions are satisfied: (i) $X|_{\Sigma_+}$ is topologically equivalent to $\widetilde{X}|_{\widetilde{\Sigma}_+}$; (ii) $Y|_{\Sigma_-}$ is topologically equivalent to $\widetilde{Y}|_{\widetilde{\Sigma}_-}$; and (iii) there is a homeomorphism $h : \Sigma \to \widetilde{\Sigma}$ such that the topological types of $Z$ at $p \in \Sigma$ and of $\widetilde{Z}$ at $\widetilde{p} = h(p) \in \widetilde{\Sigma}$ are equivalent (coincide). From this definition the concept of* **mild structural stability** *in $\Omega^r$ is naturally obtained.*

Important notions of stability in dynamical systems include that of Lyapunov (L-stability) or asymptotic stability (A-stability) at a singularity $p \in \Sigma$. Now we formalize these concepts.

**Definition 2.** *Given $p_0 \in \Sigma$ a pseudo-equilibrium of $Z \in \Omega^r$ we say that $Z$ is* **L-stable** *at $p_0$ if for all neighborhood $N_\epsilon(p_0)$ of $p_0$ in $K$ there exists a neighborhood $N_\delta(p_0)$ of $p_0$ in $K$ such that for all $p \in N_\delta(p_0)$ the future orbit of $Z$ by $p$ remains in $N_\epsilon(p_0)$.*

**Definition 3.** *Given $p_0 \in \Sigma$ a pseudo-equilibrium of $Z \in \Omega^r$ we say that $Z$ is* **A-stable** *at $p_0$ if it is L-stable and $p_0$ is the $\omega$-limit set of all $p \in N_\delta(p_0)$.*

## Main results

The main results of the paper are now stated. Theorem A establishes the local structure around $\widehat{\Omega}_1$ and Theorems B and C deal with the cumbersome task of finding conditions for A-stability of a piecewise smooth vector field presenting a cusp-fold singularity or nearby it.

**Theorem A.** It holds:
(i) $\widehat{\Omega}_1$ is a codimension one submanifold of $\Omega^r$;
(ii) If $Z \in \widehat{\Omega}_1$ then $Z$ is mild structurally stable relative to $\Omega_1$ and
(iii) $\widehat{\Omega}_1$ is open in $\Omega_1$, endowed with the topology induced from $\Omega^r$.

Denote by $\partial A$ the boundary of an arbitrary the set $A$.

**Theorem B.** Let $Z_0 = (X_0, Y_0) \in \widehat{\Omega}_1$ presenting a $Q3$-singularity $c_0$ and such that $[\varphi^+_{Y_0}((\partial\Sigma^e \cap \partial\Sigma^{c-}) \backslash \{c_0\}) \cap \Sigma] \subset \Sigma^s$ when $(X_0)_1.(Y_0)_1 < 0$. Then $Z_0$
(a) is mild structurally stable relative to $\widehat{\Omega}_1$ and
(b) is almost everywhere not A-stable in $K$.

**Theorem C.** Under the hypothesis of Theorem B consider $Z_\beta \in \Omega^r$ an unfolding of $Z_0$, where $\beta \in (-\epsilon_0, \epsilon_0)$ with $\epsilon_0 > 0$ sufficiently small. Then $Z_\beta$
(a) is mild structurally stable in $\Omega^r$ when $\beta \neq 0$ and
(b) is almost everywhere not A-stable in $K$.

When $(X_0)_1.(Y_0)_1 > 0$ we obtain the same result easier because the trajectories of $Z$ do not collide to $\Sigma$ twice.

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