Extending geometric singular perturbation theory for ordinary differential equations with three time scales

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Introduction

Systems in nature, which are modeled by ordinary differential equations (ODE), often involve two or more different time scales. For instance, in biological literature we can find many examples of models which present such features.

Example 1. The classical Rosenzweig–MacArthur predator–prey model, that is in a rescaled form given by

$$\dot{x} = x\Big(1 - x - \frac{ay}{x+d}\Big), \qquad \dot{y} = \varepsilon y\Big(\frac{ax}{x+d} - 1\Big)$$

is an example of a problem involving two different time–scales. In the above system x and y represent the number of prey and predators, respectively, the parameter $\varepsilon > 0$ is the ratio between the linear death rate of the predator and the linear growth rate of the prey. The positive parameters a and d determine the impact of predation on the prey.

Example 2. Examples of models involving three time–scales are for instance found in food chain models with a third class of so–called super or top-predators. The Rosenzweig–MacArthur model ([4]) for tritrophic food chains (as proposed by [1], see also [3])

$$\varepsilon x' = x\left(1 - x - \frac{y}{x+b_1}\right), \quad y' = y\left(\frac{x}{x+b_1} - d_1 - \frac{z}{y+b_2}\right), \quad z' = \delta z\left(\frac{y}{y+b_2} - d_2\right), \qquad (1)$$

is an example of a problem involving three different time–scales. It is composed of a logistic prey x, a Holling type II predator y and a Holling type II top–predator z.

When systems present a clear separation in time scales, methods of approximations of slow–fast systems can be applied. Around 1980, geometric singular perturbation theory was introduced. The foundation of this theory was laid by Fenichel [2] (see also [5]) and it essentially uses geometric methods from dynamical systems theory for studying the properties of solutions of the system. We note that for the singular perturbation problems studied by Fenichel only two different time–scales can be derived: a slow and a fast ones.

In this poster we study systems of ODE with three time–scales. These systems are in general written in the form

$$\varepsilon x' = f(\mathbf{x}, \varepsilon, \delta), \quad y' = g(\mathbf{x}, \varepsilon, \delta), \quad z' = \delta h(\mathbf{x}, \varepsilon, \delta), \qquad (2)$$

where $\mathbf{x} = (x, y, z) \in \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^p$, ε and δ are two independent small parameter $(0 < \varepsilon,\ 0 < \delta \ll 1)$, and f, g, h are C^r functions, where r is big enough for our purposes. Now, in system (2) three different time–scales can be derived: a slow time–scale t, an intermediate time–scale $\tau_1 := \frac{t}{\delta}$ and a fast time–scale $\tau_2 := \frac{\tau_1}{\varepsilon}$.

Note that the model (1) is a system of the form (2).

In this poster we develop a mathematical theory in order to study systems (2). Our goal is to build a theory, inspired by the one given by Fenichel in [2], for systems involving three different time–scales.

Preliminary theory

The system (2) is written with respect to the time–scale τ_1 so it is called *intermediate system*. By transforming (2) to the slow and fast variables t and τ_2 we obtain, respectively, the *slow system*

$$\varepsilon\delta\dot{x} = f(\mathbf{x}, \varepsilon, \delta), \quad \delta\dot{y} = g(\mathbf{x}, \varepsilon, \delta), \quad \dot{z} = h(\mathbf{x}, \varepsilon, \delta), \qquad (3)$$

and the *fast system*

$$\bar{x} = f(\mathbf{x}, \varepsilon, \delta), \quad \bar{y} = \varepsilon g(\mathbf{x}, \varepsilon, \delta), \quad \bar{z} = \varepsilon\delta h(\mathbf{x}, \varepsilon, \delta), \qquad (4)$$

where the dot and the bar denote the derivatives with respect to t and τ_2, respectively.
Note that, for $\varepsilon, \delta \neq 0$, systems (2), (3) and (4) are equivalent. By setting $\varepsilon = \delta = 0$ in (2), (3) and in (4) we obtain three systems with dynamics essentially different: the *intermediate problem*

$$0 = f(\mathbf{x}, 0, 0), \quad y' = g(\mathbf{x}, 0, 0), \quad z' = 0, \qquad (5)$$

the *reduced problem*

$$0 = f(\mathbf{x}, 0, 0), \quad 0 = g(\mathbf{x}, 0, 0), \quad \dot{z} = h(\mathbf{x}, 0, 0), \qquad (6)$$

the *layer problem*

$$\bar{x} = f(\mathbf{x}, 0, 0), \quad \bar{y} = 0, \quad \bar{z} = 0. \qquad (7)$$

For each ε and δ, consider the following sets

$$\mathcal{S}_1^\delta = \{\mathbf{x} \in \mathbb{R}^{n+m+p} : f(\mathbf{x}, 0, \delta) = 0\}$$

$$\mathcal{S}_2^\varepsilon = \{\mathbf{x} \in \mathbb{R}^{n+m+p} : f(\mathbf{x}, \varepsilon, 0) = g(\mathbf{x}, \varepsilon, 0) = 0\}.$$

Note that:

- the intermediate problem (5) is a dynamical system defined on $\mathcal{S}_1^0$;
- the reduced problem (6) is a dynamical system defined on $\mathcal{S}_2^0$;
- $\mathcal{S}_1^0$ is a manifold of singular points for (7).

We refer to $\mathcal{S}_1^0$ and $\mathcal{S}_2^0$ as the *intermediate and slow manifolds*, respectively. The reason for these names is that on $\mathcal{S}_1^0$ the intermediate time–scale is dominating and on $\mathcal{S}_2^0$ the slow time–scale predominates.

Following the ideas of the geometric singular perturbation theory [2], our goal will be to prove that one can obtain information on the dynamics of the system (2), for small values of ε and δ, by suitably combining the dynamics of the three limit problems (5), (6) and (7).

Four other systems will also play an important role in our analysis of system (2). By setting $\varepsilon = 0$ in (2) (or in (3)) and in (4) while keeping δ fixed but nonzero, we obtain the δ–*intermediate problem*

$$0 = f(\mathbf{x}, 0, \delta), \quad y' = g(\mathbf{x}, 0, \delta), \quad z' = \delta h(\mathbf{x}, 0, \delta), \qquad (8)$$

and the δ–*layer problem*

$$\bar{x} = f(\mathbf{x}, 0, \delta), \quad \bar{y} = 0, \quad \bar{z} = 0. \qquad (9)$$

By setting $\delta = 0$ in (2) (or in (4)) and in (3) while keeping ε fixed but nonzero, we obtain the ε–*intermediate problem*

$$\varepsilon x' = f(\mathbf{x}, \varepsilon, 0), \quad y' = g(\mathbf{x}, \varepsilon, 0), \quad z' = 0, \qquad (10)$$

and the ε–*reduced problem*

$$0 = f(\mathbf{x}, \varepsilon, 0), \quad 0 = g(\mathbf{x}, \varepsilon, 0), \quad \dot{z} = h(\mathbf{x}, \varepsilon, 0). \qquad (11)$$

Note that:

- when both $\varepsilon, \delta \to 0$, the two δ, ε–intermediate problems (8) and (10) become the same limit problem (5);
- the problem (8) is a dynamical system defined on $\mathcal{S}_1^\delta$;
- the problem (11) is a dynamical system defined on $\mathcal{S}_2^\varepsilon$;
- $\mathcal{S}_1^\delta$ is a set of singular points for the problem (9);
- $\mathcal{S}_2^\varepsilon$ is a set of singular points for the problem (10).

Definition 0.1. *We say that system* (2) *is normally hyperbolic at* $\mathbf{x}_0 \in \mathcal{S}_2^0$ *if the real parts of the eigenvalues of the Jacobian matrix*

$$\begin{pmatrix} D_{1,2}\, f(\mathbf{x}_0, 0, 0) \\ D_{1,2}\, g(\mathbf{x}_0, 0, 0) \end{pmatrix}$$

are nonzero. System (2) *is* δ*–normally hyperbolic at* $\mathbf{x}_0 \in \mathcal{S}_1^\delta$ *if the real parts of the eigenvalues of the Jacobian* $D_1 f(\mathbf{x}_0, 0, \delta)$ *are nonzero.*

Statement of the main results

We state below the main results involving systems (2).

Theorem A. *Consider the* C^r *family* (2). *Let* $\mathcal{N} \subseteq \mathcal{S}_2^0$ *be a j-dimensional compact normally hyperbolic invariant manifold of the reduced problem* (6) *Then there are* $\varepsilon_1 > 0$ *and* $\delta_1 > 0$ *and a* C^{r-1} *family of manifolds* $\{\mathcal{N}_\delta^\varepsilon : \delta \in (0, \delta_1), \varepsilon \in (0, \varepsilon_1)\}$ *such that* $\mathcal{N}_0^0 = \mathcal{N}$ *and* $\mathcal{N}_\delta^\varepsilon$ *is a hyperbolic invariant manifold of* (2).

Let $G(\mathbf{x}, \delta) := (g(\mathbf{x}, 0, \delta), \delta h(\mathbf{x}, 0, \delta), 0)$ be the vector field defined by (8) supplemented by the trivial equation $\delta' = 0$. Assume that the linearization of G at points $(\mathbf{x}, 0)$, such that $\mathbf{x} \in \mathcal{S}_2^0$, has k^s eigenvalues with negative real part and k^u eigenvalues with positive real part. The corresponding stable and unstable eigenspaces have dimensions k^s and k^u, respectively.

Similarly, let $H(\mathbf{x}, \varepsilon, \delta) := (f(\mathbf{x}, \varepsilon, \delta), \varepsilon g(\mathbf{x}, \varepsilon, \delta), \varepsilon\delta h(\mathbf{x}, \varepsilon, \delta), 0)$ be the vector field defined by (4) supplemented by the equation $\bar{\varepsilon} = 0$. Assume that the linearization of H at points $(\mathbf{x}, 0, \delta)$, such that $\mathbf{x} \in \mathcal{S}_1^\delta$, has l^s and l^u eigenvalues with negative and positive real parts, so that the corresponding stable and unstable eigenspaces have dimensions l^s and l^u, respectively.

Theorem B. *Under the hypotheses of Theorem A, suppose that* $\mathcal{N}$ *has a* $(j + j^s)$*–dimensional local stable manifold* W^s *and a* $(j + j^u)$*–dimensional local unstable manifold* W^u. *Then there are* $\varepsilon_1 > 0$ *and* $\delta_1 > 0$ *and* C^{r-1} *families of* $(j + j^s + k^s + l^s)$*–dimensional and* $(j + j^u + k^u + l^u)$*–dimensional manifolds* $\{\mathcal{W}_{\delta,\varepsilon}^s : \delta \in (0, \delta_1),\ \varepsilon \in (0, \varepsilon_1)\}$ *and* $\{\mathcal{W}_{\delta,\varepsilon}^u : \delta \in (0, \delta_1),\ \varepsilon \in (0, \varepsilon_1)\}$ *such that the manifolds* $\{\mathcal{W}_{\delta,\varepsilon}^s\}$ *and* $\{\mathcal{W}_{\delta,\varepsilon}^u\}$ *are local stable and unstable manifolds of* $\mathcal{N}_\delta^\varepsilon$, *respectively.*

Examples

Example 3. Consider the following 3–dimensional system

$$\varepsilon x' = x - \varepsilon + \delta, \quad y' = -y + \varepsilon + \delta, \quad z' = \delta z. \qquad (12)$$

The intermediate and slow manifolds $\mathcal{S}_1^0$ and $\mathcal{S}_2^0$ are given, respectively, by $\mathcal{S}_1^0 = \{(x, y, z) \in \mathbb{R}^3 : x = 0\}$ and $\mathcal{S}_2^0 = \{(x, y, z) \in \mathbb{R}^3 : x = y = 0\}$. On $\mathcal{S}_1^0$ we have defined the intermediate problem

$$0 = x, \qquad y' = -y, \qquad z' = 0, \qquad (13)$$

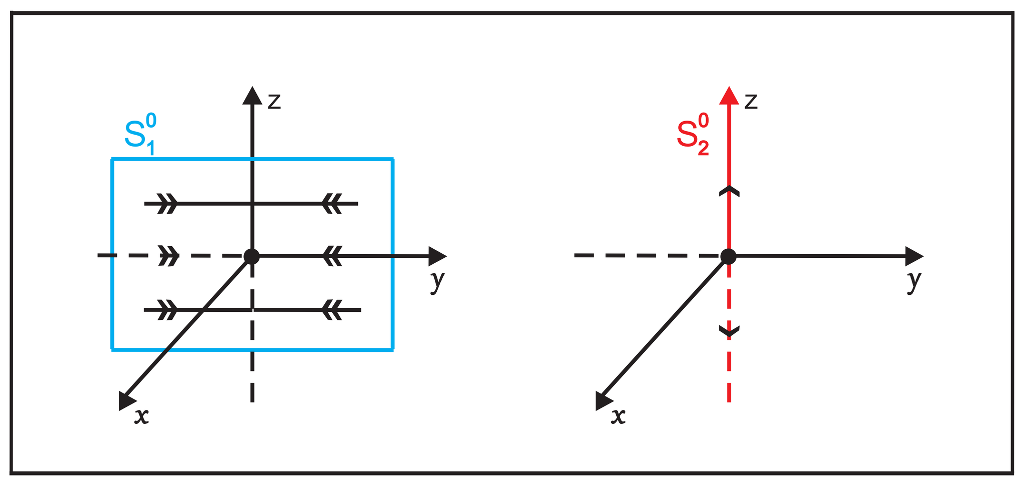
and on $\mathcal{S}_2^0$ we have defined the reduced problem

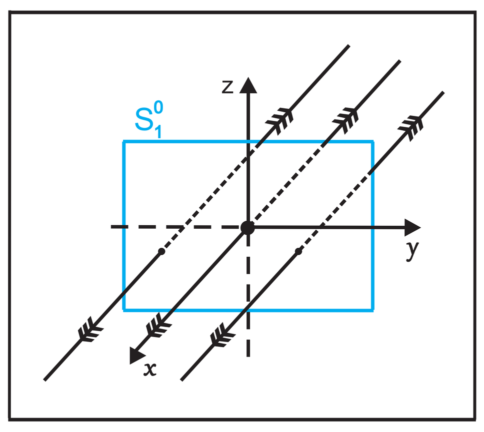
$$0 = x, \qquad 0 = y, \qquad \dot{z} = z. \qquad (14)$$

Moreover, the layer problem is given by

$$\bar{x} = x, \qquad \bar{y} = 0, \qquad \bar{z} = 0. \qquad (15)$$

Figures below illustrate the phase portraits of the intermediate, reduced and layer problems, respectively.





We can apply Theorems A and B at the normally hyperbolic singular point $\mathcal{N} = (0, 0, 0)$ of (14). Applying Theorem A, we obtain for small nonzero δ, ε, a family $\mathcal{N}_\delta^\varepsilon$ of hyperbolic singular points of (12). In fact, the family $\mathcal{N}_\delta^\varepsilon$ of singular points is given by $(\varepsilon - \delta, \varepsilon + \delta, 0)$. Applying Theorem B, we can conclude that each singular point $\mathcal{N}_\delta^\varepsilon$ has a 1–dimensional local stable manifold $\mathcal{W}_{\delta,\varepsilon}^s$ and a 2–dimensional local unstable manifold $\mathcal{W}_{\delta,\varepsilon}^u$.

Example 4. Consider the following 4–dimensional system

$$\begin{aligned} \varepsilon x' &= x - z_1 + \delta + \varepsilon = f(x, z_1, \delta, \varepsilon), \\ y' &= -y - z_2 + \delta - \varepsilon = g(y, z_2, \delta, \varepsilon), \\ z_1' &= \delta h_1(x, z_1, z_2), \\ z_2' &= \delta h_2(y, z_1, z_2), \end{aligned} \qquad (16)$$

where $h_1(x, z_1, z_2) = -z_2 - z_1(-1 + z_1^2 + z_2^2) + (x - z_1)^2$ and $h_2(y, z_1, z_2) = z_1 - z_2(-1 + z_1^2 + z_2^2) - (y + z_2)^2$. The intermediate and slow manifolds $\mathcal{S}_1^0$ and $\mathcal{S}_2^0$ are given, respectively, by $\mathcal{S}_1^0 = \{(z_1, y, z_1, z_2) \in \mathbb{R}^4 : y, z_1, z_2 \in \mathbb{R}\}$ and $\mathcal{S}_2^0 = \{(z_1, -z_2, z_1, z_2) \in \mathbb{R}^4 : z_1, z_2 \in \mathbb{R}\}$. Note that $\mathcal{S}_1^0$ and $\mathcal{S}_2^0$ are manifolds of dimension 3 and 2, respectively.
On $\mathcal{S}_1^0$ we have defined the intermediate problem

$$x = z_1, \quad y' = -y - z_2, \quad z_1' = 0, \quad z_2' = 0, \qquad (17)$$

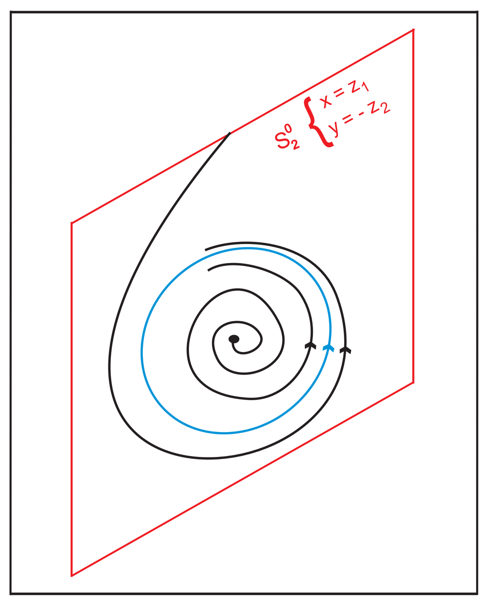
and on $\mathcal{S}_2^0$ we have defined the reduced problem

$$\dot{z}_1 = -z_2 - z_1(-1 + z_1^2 + z_2^2), \quad \dot{z}_2 = z_1 - z_2(-1 + z_1^2 + z_2^2). \qquad (18)$$

Moreover, the layer problem is given by

$$\bar{x} = x - z_1, \quad \bar{y} = 0, \quad \bar{z}_1 = 0, \quad \bar{z}_2 = 0. \qquad (19)$$

Using polar coordinates $z_1 = r\cos\theta$ and $z_2 = r\sin\theta$ it is easy to see that the system (18) presents a singular point $\mathcal{P}$ at the origin and a stable limit cycle Γ, as shown Figure below.



According with the Definition 0.1, all points of the slow manifold are normally hyperbolic. Applying Theorem A, we obtain for small nonzero δ, ε, families $\mathcal{P}_\delta^\varepsilon$ and $\Gamma_\delta^\varepsilon$ of hyperbolic singular points and limit cycles of (16), respectively, such that $\mathcal{P}_0^0 = \mathcal{P}$ and $\Gamma_0^0 = \Gamma$. In agreement with Theorem B, each singular point $\mathcal{P}_\delta^\varepsilon$ has an 1–dimensional local stable manifold $\mathcal{P}_{\delta,\varepsilon}^s$ and a 3–dimensional local unstable manifold $\mathcal{P}_{\delta,\varepsilon}^u$. Each limit cycle $\Gamma_\delta^\varepsilon$ has a 3–dimensional local stable manifold $\Gamma_{\delta,\varepsilon}^s$ and an 1–dimensional local unstable manifold $\Gamma_{\delta,\varepsilon}^u$.

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