A second order analysis of the periodic solutions for nonlinear periodic differential systems with a small parameter

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New Trends in Dynamical Systems 2012

Outline

The problem

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Examples of unperturbed systems

Systems in the standard form for averaging

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Systems in the standard form for averaging

Systems in the general form

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In the following we consider that $\mathcal{Z}$ is the image of some sufficiently smooth (C^2), one-to-one function $\xi : U \to \mathbb{R}^n$, where U is an open subset of $\mathbb{R}^k$, $1 \le k \le n$, such that $D\xi(h)$ has full rank for any $h \in U$.

Such a $\mathcal{Z}$ will be called a *T-period manifold* for (2).

The problem (Poincaré):

Study the existence of T-periodic solutions for the perturbed system ($\varepsilon \neq 0$ sufficiently small) that are "close" to the unperturbed T-periodic solutions.

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We say that a T-periodic solution $\varphi(t)$ of (2) *persists in* (1) if there exists a T-periodic solution $\varphi_\varepsilon(t)$ of (1), for small ε and $\lim_{\varepsilon \to 0} \varphi_\varepsilon(0) = \varphi(0)$.

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- for any $\varphi(t)$ with $\varphi(0) = \xi(h_0) \in \mathcal{Z}$ that persists we have that $f(h_0) = 0$;

- whenever there exists $h_0 \in U$ such that $f(h_0) = 0$ and the Jacobian determinant $\det Df(h_0) \neq 0$, the solution $\varphi(t)$ with $\varphi(0) = \xi(h_0)$ persists.

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- The planar system $x' = -y + x^2$, $y' = x + xy$ has a 2π-periodic center at the origin.
- The solution that initiates in $r_0 \in (0, 1)$ of $\frac{dr}{d\theta} = r^2 \cos\theta$ is $r(\theta, r_0) = \dfrac{r_0}{1 - r_0 \sin\theta}$ (2π-periodic for every r_0).

An autonomous system with a limit cycle

The planar system

$$\begin{array}{rcr} {x_1}' & = & x_2 - x_1(x_1^2 + x_2^2 - 1) \\ {x_2}' & = & -x_1 - x_2(x_1^2 + x_2^2 - 1) \end{array}$$

has a limit cycle: $x_1^2 + x_2^2 = 1$ of minimal period 2π. This means that

$$S^1 = \left\{(x_1, x_2) \in \mathbb{R}^2 \ : \ x_1^2 + x_2^2 = 1\right\}$$

is a 2π-period manifold of dimension 1.

The symmetric Euler top

$$\dot{x}_1 = -x_2 x_3,\ \ \dot{x}_2 = x_1 x_3,\ \ \dot{x}_3 = 0$$

This system has the following T-period manifolds (for any $m \in \mathbb{Z}$)

$$\mathcal{Z}_m^v = \{(0,0,h)\ :\ h \in (2m\pi/T, 2(m+1)\pi/T)\}$$

$$\mathcal{Z}_m^h = \left\{(z_1, z_2, 2m\pi/T)\ :\ (z_1, z_2) \in \mathbb{R}^2 \setminus \{(0,0)\}\right\}$$

A first order bifurcation function

Theorem

We consider the T-periodic, sufficiently smooth, n-dimensional system (in the standard form for averaging)

$$x'(t) = \varepsilon \tilde{F}(t, x, \varepsilon).$$

Then a first order bifurcation function is

$$f_1(z) = \int_0^T \tilde{F}(t, z, 0)dt.$$

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The idea of the proof. Let $x(t, z, \varepsilon)$ be the solution of the system satisfying $x(0, z, \varepsilon) = z$. We consider the Poincaré translation operator at time T, $z \mapsto x(T, z, \varepsilon)$ and we introduce the displacement map $\delta(z, \varepsilon) = x(T, z, \varepsilon) - z$. One can prove that

$$\delta(z, \varepsilon) = \varepsilon f_1(z) + \varepsilon^2 r(z, \varepsilon).$$

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A second order bifurcation function

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We consider the T-periodic, sufficiently smooth, n-dimensional system (in the standard form for averaging)

$$x' = \varepsilon F(t,x,\varepsilon) = \varepsilon F_1(t,x) + \varepsilon^2 F_2(t,x) + O(\varepsilon^3).$$

Then, when $f_1(z) \equiv 0$, a second order bifurcation function is

$$f_2(z) = \int_0^T F_*(t,z)dt$$

where $F_*(t,z) = F_2(t,z) + (D_z F_1(t,z)) \int_0^t F_1(t,z)dt$.

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$$\delta(z, \varepsilon) = \varepsilon f_1(z) + \varepsilon^2 f_2(z) + \varepsilon^3 r(z, \varepsilon).$$

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Hypotheses on the unperturbed system $x' = F_0(t, x)$

- (H1) $\mathcal{Z} \subset \mathbb{R}^n$ is a T-period manifold of dimension k.
- (H2) $\mathcal{Z}$ is *normally nondegenerate* (following the terminology of Carmen Chicone), that means that the linearized system around each T-periodic solution that initiates in $\mathcal{Z}$

  $$y' = D_x F_0\,(t, x(t, z, 0))\, y$$

  has the Floquet multiplier 1 of geometric multiplicity k.

Notations and facts regarding the linearized system $y' = D_x F_0\,(t, x(t,z,0))\,y \quad \text{for} \quad z \in \mathcal{Z}$

- Denote $\Phi(t,z)$ its principal matrix solution
- (H2) means that the kernel of $\Phi(T,z) - I_n$ has dimension k, or, equivalently, its range $\mathcal{R}(z)$ has dimension $n-k$
- $\mathbb{R}^n = \mathcal{R}(z) \oplus \mathcal{R}^{\perp}(z)$
- Let $\pi_C(z) : \mathbb{R}^n \to \mathcal{R}^{\perp}(z)$ be the projection
- Both the linearized system and its adjoint have exactly k linearly independent T-periodic solutions. For the adjoint system we denote them by

$$y_1(t,z),\ y_2(t,z),\ \dots\ , y_k(t,z)$$

A first order bifurcation function $M : \mathcal{Z} \to \mathbb{R}^k$

- Malkin's expression

$$M_i(z) = \int_0^T y_i(t,z) \cdot F_1(t, x(t,z,0))dt, \;\; i = \overline{1,k}$$

- Rhouma-Chicone's expression

$$M(z) = \pi_C(z)\Phi(T,z) \int_0^T \Phi^{-1}(t,z)F_1(t,x(t,z,0))dt$$

- Roseau's expression

$$M(z) = \pi \int_0^T Y^{-1}(t,z)F_1(t,x(t,z,0))dt,$$

for some *well chosen* fundamental matrix solution $Y(t,z) = \Phi(t,z)Y(0,z)$, where $\pi : \mathbb{R}^{n-k} \times \mathbb{R}^k \to \mathbb{R}^k$ is the projection onto the last k variables.

The system has the form

$$x' = F_0(t,x) + \varepsilon F_1(t,x) + \varepsilon^2 F_2(t,x) + O(\varepsilon^3)$$

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A second order bifurcation function $f_2 : U \to \mathbb{R}^k$

$$f_2(h) = 2(\pi g_2)(\xi(h)) + 2D(\pi g_1)(\xi(h))S\gamma(h) + \sum_{i=1}^{n-k} \gamma_i(h) \left[\frac{\partial}{\partial z_{k+i}} D(\pi g_0) \right] (\xi(h))S\gamma(h)$$

where

$$\gamma(h) = - \left[D(\pi^{\perp} g_0)\,(\xi(h))\,S \right]^{-1} (\pi^{\perp} g_1)(\xi(h)) \in \mathbb{R}^{n-k}$$

$$
\begin{aligned}
g_0(z) &= Y(T,z)^{-1}\left(x(T,z,0)-z\right)\\
g_1(z) &= \int_0^T Y(t,z)^{-1}F_1(t,x(t,z,0))dt\\
g_2(z) &= \frac{1}{2}\int_0^T Y(t,z)^{-1}F_*(t,x(t,z,0))dt\\
F_* &= 2F_2+2(D_xF_1)\frac{\partial x}{\partial \varepsilon}+\sum_{i=1}^{n}\frac{\partial x_i}{\partial \varepsilon}\frac{\partial}{\partial x_i}(D_xF_0)\frac{\partial x}{\partial \varepsilon}\\
\frac{\partial x}{\partial \varepsilon}(t,z,0) &= Y(t,z)\int_0^t Y(s,z)^{-1}F_1(s,x(s,z,0))ds
\end{aligned}
$$

The unperturbed system is T-isochronous: $k = n$

In this case $g_0 \equiv 0$ and the bifurcation functions have simpler expressions $f_1(z) = g_1(z)$ and $f_2(z) = 2g_2(z)$.

The idea of the proof

- we consider

$$g(z,\varepsilon) = Y(T,z)^{-1}(x(T,z,\varepsilon) - z)$$

  and note that its zeros are in one-to-one correspondence with the T-periodic solutions of our system

- we prove that $g(z,\varepsilon) = g_0(z) + \varepsilon g_1(z) + \varepsilon^2 g_2(z) + O(\varepsilon^3)$

- we apply the Lyapunov-Schmidt reduction to $g(z,\varepsilon)$

A key fact in the proof

- $g_0(z) = g(z,0) = Y(T,z)^{-1}(x(T,z,0) - z)$ and, since $\xi(h)$ is the initial value of some T-periodic solution of (2), we have that $g_0(\xi(h)) = 0$ for any $h \in U$
- $Dg_0(\xi(h)) = Y(0,\xi(h))^{-1} - Y(T,\xi(h))^{-1}$ has rank $n-k$ and its first k lines are null
- there exists some $n \times (n-k)$ matrix $S(h)$ such that the $(n-k) \times (n-k)$ matrix $\pi^{\perp} Dg_0(\xi(h))\, S(\xi)$ is invertible

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Thank you for your attention!