On the trigonometric moment problem

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Introduction

The trigonometric moment problem arises [3] from the study of one-parameter families of centers in polynomial vector fields. It asks for the classification of the trigonometric polynomials Q which are orthogonal to all powers of a trigonometric polynomial P.

We show that this problem has a simple and natural solution under certain conditions on the monodromy group of the Laurent polynomial associated to P. In the case of real trigonometric polynomials, which is the primary motivation of the problem, our conditions are shown to hold for all trigonometric polynomials of degree 15 or less. In the complex case, we show that there are a small number of exceptional monodromy groups up to degree 30 where the conditions fail to hold and show how counter-examples can be constructed in several of these cases.

1. Preliminaries

Firstly, there is an equivalence between the tangential center problem for Abel equations, the moment problem for Laurent polynomials and the vanishing of certain zero-dimensional abelian integral (see also [2]).

Proposition 1. *Let p, q be trigonometric polynomials. The following conditions are equivalent:*

1. The parametric Abel equation

$$z' = p(w)z^2 + \epsilon q(w)z^3, \tag{1}$$

has a "first order center" at the origin.

2. For P, Q the Laurent polynomials associated to primitives of p, q,

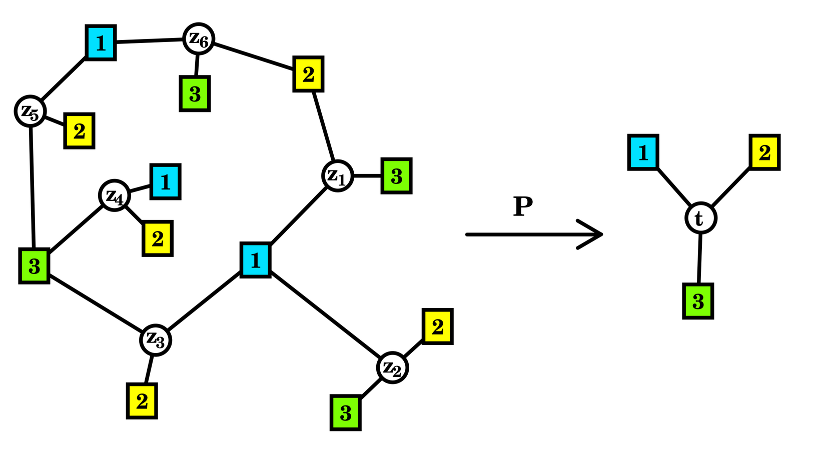
$$\oint_{|z|=1} P^k(z)\, dQ(z) = 0, \quad k = 0, 1, 2, \dots \tag{A}$$

3. Define P, Q as above, take t close to infinity and number the pre-images $z_i(t)$ of $P(z) = t$ such that $\{z_k(t)\}_{k=1,\dots,n}$ (resp. $\{z_k(t)\}_{k=n+1,\dots,n+m}$) are the points close to infinity (resp. zero). Then

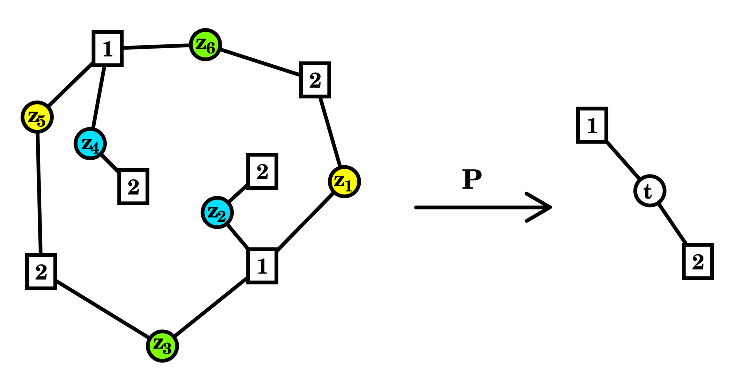
$$\sum_{k=1}^{n} mQ(z_k(t)) - \sum_{k=n+1}^{n+m} nQ(z_k(t)) \equiv 0 \quad \text{for every } t \in \mathbb{C}. \tag{2}$$

Let $P \in \mathcal{L}$ be a Laurent polynomial, let $G_P = Gal(L/\mathbb{C}(t))$ denote its monodromy group, where $L = \mathbb{C}(z_1(t), \dots, z_{n+m}(t))$ and $z_1, \dots, z_{n+m}$ are the branches of P^{-1}. We shall number the branches of P^{-1} such that $\sigma_\infty = (1, 2, \dots, n)(n+1, n+2, \dots, n+m)$ is a permutation corresponding to a clockwise loop around infinity.

A useful way to represent the monodromy group G_P of P is to take the set of critical values, $\Sigma = \{t_1, t_2, \dots, t_r\}$, and fix t_0 a non-critical value, then consider the graph obtained as a pre-image of the "star" obtained by joining each of the critical values to t_0 with non-intersecting paths.



Assume that $P = \tilde{P} \circ W$. Consider the partition into disjoint sets induced in $\{1, \dots, n+m\}$ by the equivalence $i \sim j$ whenever $W(z_i(t)) = W(z_j(t))$. This partition is an imprimitivity system and the elements are called blocks. It holds that the monodromy group sends a block to itself or to another block. Moreover, if a partition of $\{1, \dots, n+m\}$ into disjoint sets satisfy this property, then there exist $\tilde{P}, W$ such that $P = \tilde{P} \circ W$ and W is constant on the elements of the partition.



A Laurent polynomial P can be written as $\tilde{P}(z^n)$, where $\tilde{P} \in \mathcal{L}$, or as $\tilde{P}(W)$, where $\tilde{P} \in \mathbb{C}[z]$ and $W \in \mathcal{L}$ (see [4] or [7]).

2. Main Results

Firstly we provide some sufficient conditions such that (A) holds.

Proposition 2. *Suppose that*

$$P = \tilde{P}(z^l), \quad Q = \tilde{Q}(z^l) + \sum_{l \nmid i} a_i z^i, \qquad \oint_{|z|=1} \tilde{P}^k(z)\, d\tilde{Q}(z) = 0, \quad k = 0, 1, 2, \dots, \tag{B}$$

then (A) holds. Conversely, if (A) holds and $P = \tilde{P}(z^l)$, $Q = \tilde{Q}(z^l) + \sum_{l \nmid i} a_i z^i$, then

$$\oint_{|z|=1} \tilde{P}^k(z)\, d\tilde{Q}(z) = 0, \quad k = 0, 1, 2, \dots.$$

That is, when $P = \tilde{P}(z^l)$ we can reduce the problem. We determine this type of decompositions in terms of the monodromy group: $P = \tilde{P}(z^l)$ if and only if

$$\exists \mathcal{B} \in \mathbb{B} \quad \exists B \in \mathcal{B}, \qquad \{1\} \subsetneq B \subseteq \{1, \dots, n\}, \tag{B^*}$$

where $\mathbb{B}$ denote the set of all imprimitivity systems.

Proposition 3. *Suppose that*

$$P = P_k(W_k), \quad P_k, Q_k \in \mathbb{C}[z], \quad W_k \in \mathcal{L}, \quad k = 1, \dots, l, \qquad Q = Q_1 \circ W_1 + \dots + Q_l \circ W_l, \tag{C}$$

then (A) holds.

Now we want to determine when (C) is also sufficient in terms of the action of the monodromy group. Let

$$V = <\sigma(m, \overset{(n)}{\dots}, m, -n, \overset{(m)}{\dots}, -n) : \sigma \in G_P>,$$

where G_P acts permuting the coordinates.

For each block B we define w_B to be the vector with $(w_B)_i = 1$ when $i \in B$ and 0 otherwise. We define W to be the space generated by all vectors w_B where B runs over all blocks which contain the element 1, including the trivial block $B = \{1, \dots, n+m\}$, but not $\{1\}$.

Theorem 4. *Let P be a proper Laurent Polynomial ($P \notin \mathbb{C}[z]$, $P \notin \mathbb{C}[z^{-1}]$), such that it does not admit a decomposition of the form $P(z) = \tilde{P}(z^l)$ for any $l > 1$. Assume that*

$$(1, 0, \dots, 0) \in V + W. \tag{C^*}$$

Then (A) is equivalent to (C).

3.Computations up to degree 30

For a given degree of P there is a finite number of possible monodromy groups. We check when (B^*) and (C^*) do not apply:

Degree	9	10	16	18	20	24	25	27	30
Groups	34	45	1954	983	1117	25000	211	2392	5712
Exceptions	1	2	6	6	3	3	2	31	10

Theorem 5. *For any Laurent polynomial P up to degree 30, if G_P is not one of the exceptional groups in the list above, then $Q \in \mathcal{L}$ satisfies (A) if and only if it is reducible via condition (B) to a set of moment equations of lower degree, or satisfies the weak composition condition (C).*

Each of the exceptional groups above can be realized as the monodromy group of a Laurent polynomial, although it is only possible to explicitly calculate these polynomials in simple cases. We have not been able to verify that each exceptional group does indeed give Laurent polynomials P and Q which satisfy (A) but neither (B) nor (C), but below we give examples of three of the four simplest cases of exceptional groups, showing how they indeed give such Q. A fourth case is considered by Pakovich, Pech and Zvonkin [6].

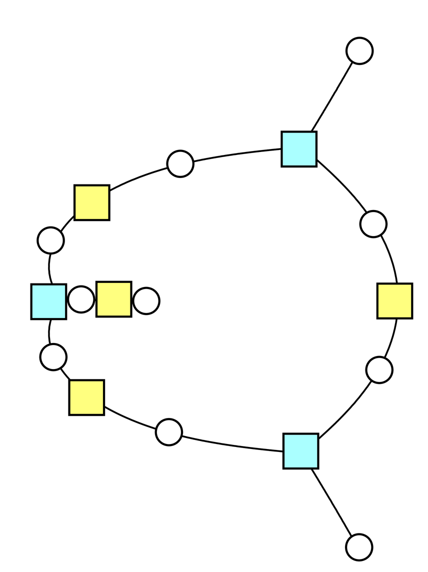
Group $A_5(10)$

This group is realized by the following indecomposable Laurent polynomial:

$$P(z) = \frac{(z+d)R^3(z)}{z^5},$$

where

$$R(z) = \frac{(123 + 55\sqrt{5})}{2} + \frac{(29 + 13\sqrt{5})z}{2} - (2 + \sqrt{5})z^2 + z^3.$$

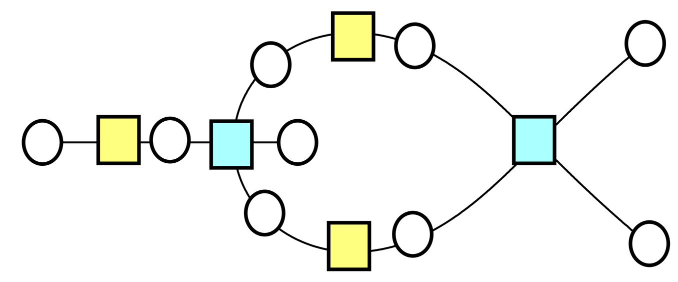


There is a 4-dimensional space of Laurent polynomials Q of degree lower that P satisfying (A), where (B) and (C) does not hold.

Group $E9 : D_8$

The group is realized by the indecomposable polynomial

$$P(z) = -\frac{(z-1)^4(2+z)(1+2z)^4}{2z^3}.$$



There is a 3-dimensional space of Laurent polynomials Q of degree lower that P satisfying (A), where (B) and (C) does not hold.

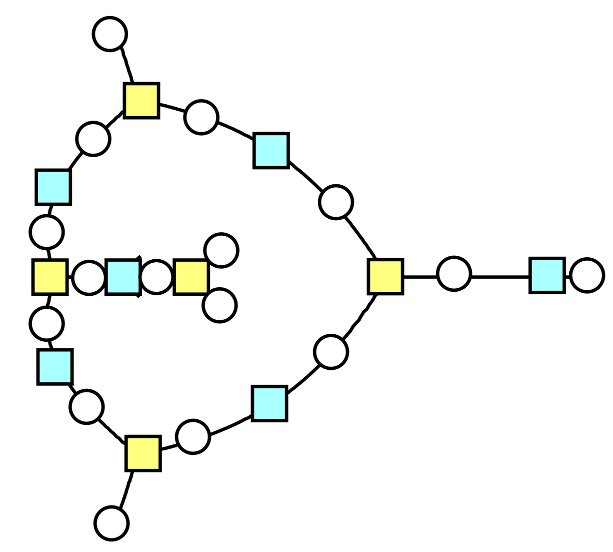
Group $t16n195$

The group is realized by the composed Laurent polynomial

$$P(z) = -940848 + 665280\sqrt{2} + (89152 - 63040\sqrt{2})W(z) - (2376 - 1680\sqrt{2})W^2(z) + W^4(z),$$

where the composition factor is

$$W(z) = \frac{-99 + 70\sqrt{2} + (48 - 34\sqrt{2})z + (8 - 6\sqrt{2})z^3 + z^4}{z^2}.$$



The solutions form a 6-dimensional space, which is bigger that the dimension of the polynomials satisfying (B) or (C).

4. Real case

Finally, we study when the trigonometric polynomials have real coefficients. Then the condition $\tilde{P}(z_i(t)) = \overline{\tilde{P}(1/\overline{z_i(t)})}$ holds for real t , and this allows us to assume that the monodromy group contains the dihedral group.

Theorem 6. *Let $\mathcal{L}_{30}^*$ be the set of Laurent polynomials up to degree 30 associated to real trigonometric polynomials via $z = \exp(i\theta)$. For any $P \in \mathcal{L}_{30}^*$, $Q \in \mathcal{L}$ satisfies (A) if and only if either it is reducible via condition (B) to a set of moment equations of lower degree, or it is satisfied condition (C). We can iterate the process of reduction of (B) so that all Q satisfying (A) can be explained by these two processes.*

Conjecture 7. *For every Laurent polynomial P obtained by a change of variables from a real trigonometric polynomial, a Laurent polynomial Q satisfies (A) if and only if it satisfies (B) or (C).*

Theorem 8. *Suppose that $deg(P) = 2n > 2$, where n is a prime number. Then $Q \in \mathcal{L}$ satisfies (A) if and only if one of the following possibilities holds:*

- *$P(z) = \tilde{P}(z^n)$, and*

$$\oint_{|z|=1} \tilde{P}^k(z)\, d\tilde{Q}(z) = 0, \quad k = 0, 1, 2, \dots$$

where $\tilde{Q}(z^n)$ is the sum of the monomials of Q divisible by z^n.

- *There exist $W \in \mathcal{L} \setminus (\mathbb{C}[z] \cup \mathbb{C}[z^{-1}])$, $\tilde{P}, \tilde{Q} \in \mathbb{C}[z]$, such that $P(z) = \tilde{P}(W(z))$, $Q(z) = \tilde{Q}(W(z))$.*

Thus, (A) holds if and only if (B) or (C) holds.

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