Semiconjugacy to a map of a constant slope - new results

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Topological entropy

X... c.m.sp., $f\colon\ X \to X$ continuous

- $E \subset X$ is (n, ε)-separated (with respect to f) if

$$\forall\, x, y \in E,\ x \neq y\colon \quad \max_{0 \leq i \leq n-1} d(f^i(x), f^i(y)) > \varepsilon$$

- $s(n, \varepsilon)$ is the largest cardinality of any (n, ε)-separated subset of X (it is finite)

The topological entropy $h_{top}(f)$ of a map f is the quantity

$$\lim_{\varepsilon \to 0_+} \limsup_{n \to \infty} \frac{1}{n} \log s(n, \varepsilon)$$

Let us consider continuous maps $f\colon\ X \to X$ and $g\colon\ Y \to Y$, where X, Y are compact Hausdorff spaces and $\varphi\colon\ X \to Y$ is continuous such that the diagram

$$\begin{array}{ccc} X & \xrightarrow{f} & X \\ \downarrow\varphi & & \downarrow\varphi \\ Y & \xrightarrow{g} & Y \end{array}$$

commutes, i.e., $\varphi \circ f = g \circ \varphi$. When φ is surjective, we say that f is semiconjugated to g via a map φ and in that case the topological entropy $h_{top}(\cdot)$ satisfies $h_{top}(f) \geq h_{top}(g)$.

M. Misiurewicz, W. Szlenk, *E*ntropy of piecewise monotone mappings, Studia Math. 67(1) (1980), 45-63.

A continuous map $f\colon\ [0,1] \to [0,1]$ is said to be piecewise monotone if there are $k \in \mathbb{N}$ and points $0 = c_0 < c_1 < \cdots < c_{k-1} < c_k = 1$ such that f is monotone on each $[c_i, c_{i+1}]$, $i = 0, \ldots, k-1$. We shall say that a piecewise monotone map g has a constant slope s if on each of its pieces of monotonicity it is affine with the slope of absolute value s.

Theorem

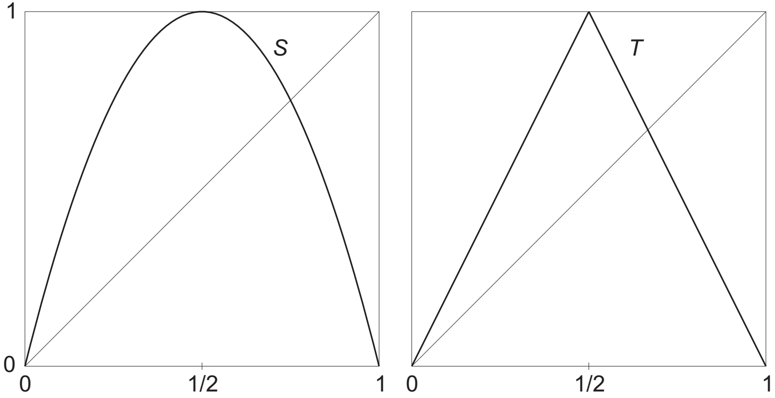
- *If $h(f) > 0$ then $h(f) = \lim_n \frac{1}{n} \log \ell(f^n)$, $\ell(f^n)$ denotes the number of pieces of monotonicity of f^n.*
- *It is known that if g has a constant slope s then $h_{top}(g) = \max(0, \log s)$.*

W. Parry, *Symbolic dynamics and transformations of the unit interval*, Trans. Amer. Math. Soc. 122 (1966), 368–378.

J. Milnor, W. Thurston, *On iterated maps of the interval*, Dynamical Systems, 465-563, LNM 1342, Springer, Berlin, 1988.

Theorem

(Parry 66; Milnor, Thurston 88) If f is piecewise monotone and $h_{top}(f) > 0$ then f is semiconjugated via a continuous non-decreasing map to some map g of constant slope $e^{h_{top}(f)}$ (conjugated when f is transitive).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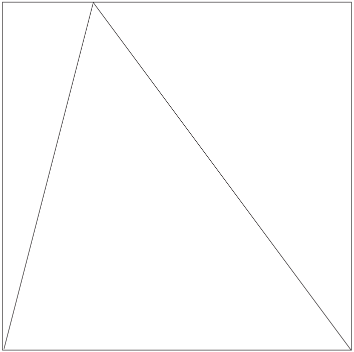
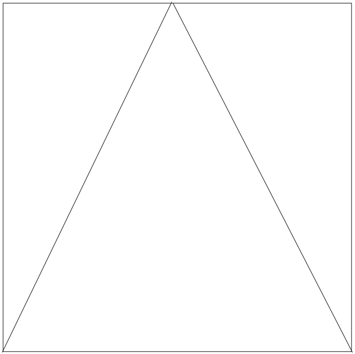




The conjugacy is not absolutely continuous.

We focus on the class of Markov *countably* piecewise monotone continuous interval maps and try to find large subclass(es) of it in which the conclusion of Theorem remains true.

an *admissible* set P . . . *finite* or *countably infinite* closed subset of $[0,1]$ containing the points $0, 1$

an interval $[a,b] \subset [0,1]$ is P-basic . . . $a, b \in P$ and $(a,b) \cap P = \emptyset$

$B(P)$ . . . the set of all P-basic intervals

a continuous $f\colon\ [0,1] \to [0,1]$ is in the class $\mathcal{CPM}$ if and only if it corresponds to some admissible set P such that

- $f\colon\ P \to P$
- f is monotone (perhaps constant) on each P-basic interval

A map $f \in \mathcal{CPM}$ which is not piecewise monotone will be called a *countably piecewise monotone map*.

For P admissible we denote

$\mathcal{M}_P$ . . . the set of all (possibly generalized, multi-infinite) matrices indexed by P-basic intervals and with entries from $[0, \infty]$

ℓ^1_P . . . the Banach space of all real absolutely convergent (again possibly multi-infinite) sequences indexed by P-basic intervals

$\mathcal{K}^+_P$ . . . the cone of all nonnegative sequences from ℓ^1_P

Remark

For an admissible set P, a matrix $M \in \mathcal{M}_P$ can be modeled as a table $(P \times [0,1]) \cup ([0,1] \times (1-P))$; an entry of M is a number from $[0, \infty]$ in one window indexed IJ, where $I \in B(1-P)$ and $J \in B(P)$. Let us denote P' the set of all limit points of P. In accordance with the above model, a matrix $M \in \mathcal{M}_P$ will be infinite in the usual sense if $P' = \{1\}$. We call it multi-infinite when card$P' > 1$. For example, for the choice $P = \{0\} \cup \{\frac{1}{2^m} + \frac{1}{2^n}\}_{m,n \geq 1}$ we get card$P' = \infty$.

Definition

For an $f \in \mathcal{CPM}$ we define its matrix $M(f) \in \mathcal{M}_P$: the m_{IJ} entry of $M(f)$ is 1 if $f(I) \supset J$, and 0 otherwise.

In general, for f from $\mathcal{CPM}$ its matrix $M(f)$ does not represent a bounded operator on ℓ^1_P.

Proposition

Let $M = (m_{IJ}) \in \mathcal{M}_P$. Then

(i) *M represents a bounded linear operator $\mathbb{M}$ on the ℓ^1_P defined as*

$$(\mathbb{M}u)_I := \sum_{J \in B(P)} m_{IJ} u_J, \ u \in \ell^1_P, \tag{1}$$

if and only if $(\|\mathbb{M}\| =) \ \sup_{J \in B(P)} \sum_{I \in B(P)} |m_{IJ}| < \infty$. In that case the operator $\mathbb{M}$ is $\mathcal{K}^+_P$-positive.

(ii) *The operator $\mathbb{M}$ is compact if and only if its representing matrix M satisfies*

$$\forall \varepsilon > 0 \ \exists \delta \ \forall J \in B(P) \colon \ \sum |m_{IJ}| < \varepsilon.$$

General observation - J. B., *Semicojugacy to a map of a constant slope*, Studia Mathematica 208 (2012), 213-228.

$\mathcal{CPM}_\lambda$ . . . the class of all maps from $\mathcal{CPM}$ of a constant slope λ, i.e., $f \in \mathcal{CPM}_\lambda$ if $|f'(x)| = \lambda$ for all $x \in [0,1]$, possibly except at the points of P

Theorem - Key Equation

Let $f \in \mathcal{CPM}$ with $M(f) = (m_{IJ}) \in \mathcal{M}_P$. Then f is semiconjugated via a continuous non-decreasing map ψ to some map $g \in \mathcal{CPM}_\lambda$, $\lambda > 1$, if and only if there is a nonzero vector $v = (v_I)_{I \in B(P)}$ from $\mathcal{K}_P^+$ such that

$$\forall\ I \in B(P): \quad \sum_{J \in B(P)} m_{IJ} v_J = \lambda\ v_I. \tag{2}$$

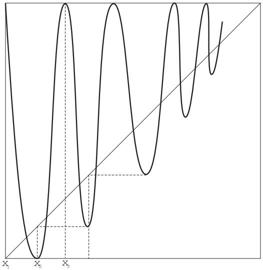
We will need a genealogic tree $(P_n)_{n=0}^{\infty}$ of P with respect to f.
We set $P_0 = P$. By the previous, f is not constant on any P_0-basic interval.
Suppose that P_n is already defined and f is not constant on any P_n-basic intervals. Since f is countably piecewise monotone, $f^{-1}(P_n) \cap [0,1]$ is a union of a (at most) countably many closed intervals (perhaps degenerate). Since f was not constant on any P_n-basic interval, no component of $f^{-1}(P_n) \cap [0,1]$ contains more than one element of P_n. From each of these components we choose one point; if possible the element of P_n, and we define P_{n+1} to be the set of these chosen points. Thus $P_n \subset P_{n+1}$ and P_{n+1} is invariant since $f(P_{n+1}) \subset P_n$. By the construction, P_{n+1} is a countable set and f is not constant on any P_{n+1}-basic interval.
Denote $\mathcal{J}_n$ the set of all P_n-basic intervals. In particular, $\mathcal{J}_0 = B(P)$. Let $v = (v_I)_{I \in B(P)} \in \mathcal{K}_P^+$ be a normalized vector satisfying (2).
In order to define the map $\psi\colon\ Q = \bigcup_{n=0}^{\infty} P_n \to [0,1]$ we put $\psi(0) = 0$ and for $x \in P_n \cap (0,1]$

$$\psi(x) = \lambda^{-n} \sum_{J \in \mathcal{J}_n,\ J \leq x} v_{f^n(J)}. \tag{3}$$

Example

In order to illustrate the Key Equation let $P = \{1\} \cup \{x_n = 1 - \frac{1}{n}\}_{n \geq 1}\}$ with P-basic intervals $I(n) = [x_n, x_{n+1}]$ and consider a map f from $\mathcal{CPM}$ such that $f(x_2) = x_1 = 0$ and

$$f(x_n) = \begin{cases} 1, & n \geq 1 \text{ odd}, \\ x_{n-2}, & n \geq 4 \text{ even}. \end{cases}$$



The Key Equation has a solution $v = (v_{I(n)})_{I(n) \in B(P)}$ with $v_{I(2k+1)} = v_{I(2k+2)} = \frac{k+1}{\lambda}(\frac{\lambda-1}{2\lambda})^k$, $k \geq 0$ for any $\lambda \geq 3+\sqrt{8}$. In particular, the map f is semiconjugated via a non-decreasing map ψ to some map $g \in \mathcal{CPM}_{3+\sqrt{8}}$ (in fact one can show that $h_{top}(f) = \log(3+\sqrt{8})$).

The matrix $M(f)$ of f does not represent a bounded linear operator on the space ℓ^1_P.

$$M(f) = \left(\begin{array}{cccccccccccccccccccc}
1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&\cdot&\cdot&\cdot&\cdot\\
1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&\cdot&\cdot&\cdot&\cdot\\
0&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&\cdot&\cdot&\cdot&\cdot\\
0&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&\cdot&\cdot&\cdot&\cdot\\
0&0&0&1&1&1&1&1&1&1&1&1&1&1&1&1&\cdot&\cdot&\cdot&\cdot\\
0&0&0&1&1&1&1&1&1&1&1&1&1&1&1&1&\cdot&\cdot&\cdot&\cdot\\
0&0&0&0&0&1&1&1&1&1&1&1&1&1&1&1&\cdot&\cdot&\cdot&\cdot\\
0&0&0&0&0&1&1&1&1&1&1&1&1&1&1&1&\cdot&\cdot&\cdot&\cdot\\
0&0&0&0&0&0&0&1&1&1&1&1&1&1&1&1&\cdot&\cdot&\cdot&\cdot\\
0&0&0&0&0&0&0&1&1&1&1&1&1&1&1&1&\cdot&\cdot&\cdot&\cdot\\
0&0&0&0&0&0&0&0&0&1&1&1&1&1&1&1&\cdot&\cdot&\cdot&\cdot\\
0&0&0&0&0&0&0&0&0&1&1&1&1&1&1&1&\cdot&\cdot&\cdot&\cdot\\
0&0&0&0&0&0&0&0&0&0&0&1&1&1&1&1&\cdot&\cdot&\cdot&\cdot\\
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\end{array}\right)$$

The Key Equation has a solution for each $\lambda \geq 3+\sqrt{8}$.

Example

$V=\{v_i\}_{i\geq -1}$, $X=\{x_i\}_{i\geq 1}$ V,X converge to $1/2$ and
$0=v_{-1}=x_0=v_0<x_1<v_1<x_2<v_2<x_3<v_3<\cdots$

$$f=f(V,X):[0,1]\to[0,1]$$

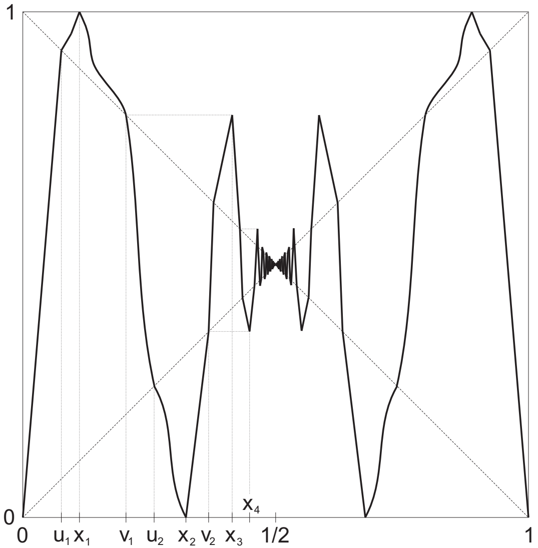
(a) $f(v_{2i-1})=1-v_{2i-1}$, $i\geq 1$, $f(v_{2i})=v_{2i}$, $i\geq 0$,

(b) $f(x_{2i-1})=1-v_{2i-3}$, $i\geq 1$, $f(x_{2i})=v_{2i-2}$, $i\geq 1$,

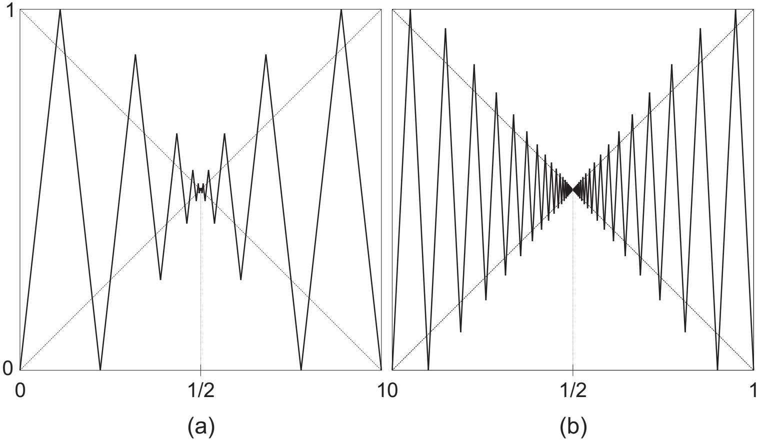
(c) $f_{u,v}=\left|\frac{f(u)-f(v)}{u-v}\right|>1$ for each interval $[u,v]\subset[x_i,x_{i+1}]$,

(d) $f(1/2)=1/2$ and $f(t)=f(1-t)$ for each $t\in[1/2,1]$.

(the property (c) can be satisfied since for our V,X by (a),(b), $f_{x_i,x_{i+1}}>2$ for each $i\geq 0$)
We denote by $\mathcal{F}(V,X)$ the set of all continuous interval maps fulfilling (a)-(d) for a fixed pair V,X and $\mathcal{F}:=\bigcup_{V,X}\mathcal{F}(V,X)$.



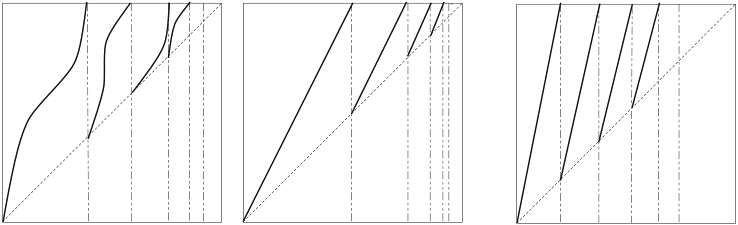
The Key Equation is solvable for each $\lambda \geq 9$.



(a) The map f_9; (b) the map f_{20}.

Example

In the Key Equation we do not assume that the entropy of f is positive, but $\lambda > 1$ only. For example, all three maps have the same matrix $M \in \mathcal{M}_P$; the corresponding Key Equation is solvable for any $\lambda > 1$ with the formula $v_j = (1 - \frac{1}{\lambda})^j$, $j \geq 0$.



(a) The maps from one conjugacy class.
The Key Equation is solvable for any $\lambda > 1$.

$$M = \begin{pmatrix}
\mathbf{1} & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & \cdot & \cdot & \cdot & \cdot \\
0 & \mathbf{1} & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & \cdot & \cdot & \cdot & \cdot \\
0 & 0 & \mathbf{1} & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & \cdot & \cdot & \cdot & \cdot \\
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\end{pmatrix}$$

Two cases

For $f \in \mathcal{CPM}$,

1. either $M(f)$ represents a bounded linear operator on ℓ^1_P and we need to use extensions of Perron-Frobenius theorem for positive operators leaving invariant a cone in a real Banach space;

2. or $M(f)$ does not represent a bounded linear operator on ℓ^1_P and we need to use the Vere-Jones classification of infinite positive matrices.

Case of bounded operators

In order to use our theorem effectively we restrict our attention to a (still sufficiently rich) subclass of maps from $\mathcal{CPM}$.
To this end let us denote $\mathcal{P}$ the set of all pairs (P, φ) such that

(A1) P is admissible,

(A2) $\varphi\colon\ P \to P$ is continuous,

(A3) continuous 'connect-the-dots' map $\varphi_P\colon\ [0,1] \to [0,1]$ defined by $\varphi_P|_P = \varphi$, $\varphi_P|_J$ affine for any interval $J \subset \mathrm{conv}(P)$ such that $J \cap P = \emptyset$, satisfies

$$\exists L = L(P, \varphi) > 0\ \forall I \in B(P)\ \forall y \in I^\circ\colon\ \mathrm{card}\varphi_P^{-1}(y) < L.$$

In this part we will deal with *restrictively countably piecewise monotone continuous maps* from the class $\mathcal{RCPM}$, where $f \in \mathcal{RCPM}$ if and only if it corresponds to some pair $(P, \varphi) \in \mathcal{P}$, i.e., $f|P = \varphi$ and f is monotone on each P-basic interval.

Proposition

Let $M(f) \in \mathcal{M}_P$ be the matrix of $f \in \mathcal{RCPM}$. Then $M(f)$ represents a bounded $\mathcal{K}_P^+$-positive linear operator on ℓ_P^1.

Up to now we do not have any information on the relationship of the entropy of $f \in \mathcal{RCPM}$ and its spectral radius $r(\mathbb{M})$. This gap will be partially filled in by the following theorem.

Theorem

Let $M(f) \in \mathcal{M}_P$ be the matrix of $f \in \mathcal{RCPM}$, denote $\mathbb{M}$ the operator on ℓ_P^1 represented by $M(f)$ and assume that $h_{top}(f) > 0$. Then $r(\mathbb{M}) \geq e^{h_{top}(f)}$.

S. Karlin, *P*ositive operators, J. Math. Mech. 8 (1959), 907–937.

$\mathcal{K}_P^+$ is reproducing: $\mathcal{K}_P^+ - \mathcal{K}_P^+ = \ell_P^1$
$\mathcal{K}_P^+$ defines a partial ordering on ℓ_P^1: $x \leq y$ if $y - x \in \mathcal{K}_P^+$
$\mathcal{K}_P^+$ is normal:
$\exists b > 0 \; \forall x, y \in l_P^1$: $\theta \leq x \leq y \implies ||x|| \leq b||y||$ (even acute, $b = 1$)

For a bounded linear operator $\mathbb{A}$ on a Banach space $\mathcal{E}$ we will consider its spectrum $\sigma(\mathbb{A}) = P_\sigma(\mathbb{A}) \cup R_\sigma(\mathbb{A}) \cup C_\sigma(\mathbb{A})$ partitioned to the point, residual and continuous part respectively.

Theorem

(Krejn-Bonsall-Karlin) Let $\mathcal{K}$ be a normal reproducing cone in a (real) Banach space $\mathcal{E}$. Then, for every bounded positive operator $\mathbb{A}$ ($\mathbb{A}\mathcal{K} \subset \mathcal{K}$), the spectral radius $r(\mathbb{A})$ of $\mathbb{A}$ belongs to the spectrum.

I. Marek, *O*n some spectral properties of Radon-Nicolski operations and their generalizations, Comm. Math. Univ. Carolinae 3 (1962), 20–30.

Definition

A bounded linear operator $\mathbb{A}$ defined on a (complex) Banach space $\mathcal{F}$ will be called Radon-Nicolski if $\mathbb{A}$ may be represented as $\mathbb{A} = \mathbb{C} + \mathbb{B}$, where

(i) $\mathbb{C}$ is compact,

(ii) $r(\mathbb{A}) > r(\mathbb{B})$.

Theorem

Let τ be a function holomorphic in the neighborhood of the spectrum $\sigma(\mathbb{A})$ of the operator $\mathbb{A}$. Let $\mathbb{A}$ be $\mathcal{K}$-positive and assume that $\tau(\mathbb{A})$ is a Radon-Nicolski operator on a real Banach space $\mathcal{E}$. Then $r(\mathbb{A}) \in P_\sigma(\mathbb{A})$ with corresponding eigenvector in $\mathcal{K}$.

The previous results lead us to the following results:

Theorem

Let $M(f) \in \mathcal{M}_P$ be the matrix of $f \in \mathcal{RCPM}$, denote $\mathbb{M}$ the operator on ℓ^1_P represented by $M(f)$ and assume that $h_{top}(f) > 0$. If $\tau(\mathbb{M})$ is a Radon-Nicolski operator on ℓ^1_P for a suitable τ holomorphic in the neighborhood of the spectrum $\sigma(\mathbb{M})$, then
- $r(\mathbb{M}) = e^{h_{top}(f)} = \beta$
- f is semiconjugated via a non-decreasing map ψ to some map $g \in \mathcal{RCPM}_\beta$; in particular it is true when $\mathbb{M}$ itself is a Radon-Nicolski operator

Definition

A function $f\colon\ [a,b] \to [a,b]$ is said to have a *d-horseshoe* if there exist d subintervals $I_1, I_2, \ldots, I_d$ of $[a,b]$ with disjoint interiors such that $f(I_i) \supset I_j$ for all $1 \leq i, j \leq d$.

Definition

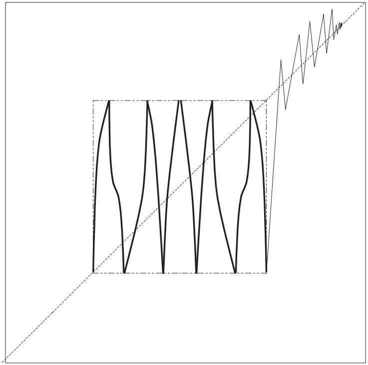
For an integer $m > 1$, we say that a pair $(P, \varphi) \in \mathcal{P}$ is m-ruled, if there are P-basic intervals $I_1, \ldots, I_m$ such that

- $\varphi_P\colon\ [0,1] \to [0,1]$ has an m-horseshoe created by the intervals $I_1, \ldots, I_m$
- $\forall I \in B(P)\ \forall y \in I^\circ\colon\ \operatorname{card}[\varphi_P^{-1}(y) \cap ([0,1] \setminus \bigcup_{i=1}^m I_i)] < m.$

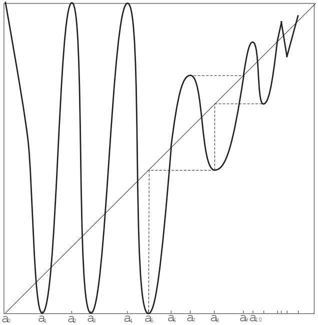
Theorem

Let $f \in \mathcal{RCPM}$ correspond to an m-ruled pair $(P, \varphi) \in \mathcal{P}$. Then f is semiconjugated via a non-decreasing map ψ to some map $g \in \mathcal{RCPM}_\beta$ with $\beta = e^{h_{top}(f)}$.

Example



Sketch of a map corresponding to a 10-ruled pair from $\mathcal{P}$.



$f \in \mathcal{RCPM}$ transitive (left), $g \in \mathcal{RCPM}_{r(\mathbb{M})}$,
$r(\mathbb{M}) = e^{h_{top}(f)} \sim 6.5616.$

Example

Let $P = \{a_n\colon\ n = 0, 1, \ldots, \infty\}$ be an admissible set with the only limit point equal to 1. Assume that $0 = a_0 < a_1 < \cdots < a_\infty = 1$ and define the map $\varphi\colon\ P \to P$ by $\varphi(a_0) = \varphi(a_2) = \varphi(a_4) = a_\infty$, $\varphi(a_1) = \varphi(a_3) = \varphi(a_5) = a_0$, $\varphi(a_6) = a_6$ and for each $k \geq 0$, $\varphi(a_{3k+7}) = a_{3k+9}$, $\varphi(a_{3k+8}) = a_{3k+5}$, $\varphi(a_{3k+9}) = a_{3k+9}$. Then $(P, \varphi) \in \mathcal{P}$ and it is 6-ruled, where φ_P has a 6-horseshoe created by the P-basic intervals $[a_0, a_1], \ldots, [a_5, a_6]$. Let us consider a map $f \in \mathcal{RCPM}$ that corresponds to (P, φ). By our theorem the map f is semiconjugated via a non-decreasing map ψ to some map $g \in \mathcal{RCPM}_\beta$ with $\beta = e^{h_{top}(f)}$. In particular, the maps f, g are conjugated when f is transitive.

The matrices $M(f), C, B$ from Example:

$$M(f) = \begin{pmatrix}
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\end{pmatrix}.
$$

Case of unbounded operators

Theorem

Let $T = (t_{ij})$ be an infinite nonnegative irreducible aperiodic matrix. There exists a common value $\lambda_M = \frac{1}{R}$ such that for each i, j

$$\lim_n (t_{ij}^{(n)})^{\frac{1}{n}} = \lambda_M.$$

Theorem

For any value $r > 0$ and all i, j

(i) *the series $\sum_n t_{ij}^{(n)} r^n$ are either all convergent or all divergent;*

(ii) *as $n \to \infty$, either all or none of the sequences $\{t_{ij}^{(n)} r^n\}_n$ tend to zero.*

D. Vere-Jones, *Ergodic properties of non-negative matrices-I*, Pacific J. of Math. **22**(2)(1967), 361–385.

Definition

(i) The matrix $T = (t_{ij})$ is R-transient or R-recurrent according as the series $\sum_n t_{ij}^{(n)} R^n$ are convergent or divergent;

(ii) an R-recurrent matrix is R-null or R-positive according as all or none of the sequences $\{t_{ij}^{(n)} R^n\}_n$ tend to zero.

J. B., H. Bruin, *Semiconjugacy to a map of a constant slope II*, in preparation.

Theorem

Let $M(f) \in \mathcal{M}_P$ be the matrix of $f \in \mathcal{CPM}$ which is R-transient. Then the Key Equation does not have any solution.

Theorem

There exists a map $f \in \mathcal{CPM}$ such that $M(f) \in \mathcal{M}_P$ is R-null with $R \in (0,1)$ and the Key Equation does not have any solution.

Theorem

For $f \in \mathcal{CPM}$ with $h_{top}(f) > 0$, assume that f^n has a full lap for some n. If $M(f) \in \mathcal{M}_P$ is R-positive, then $R \in (0,1)$ and f is semiconjugated via a non-decreasing map ψ to some map $g \in \mathcal{CPM}_\beta$, where $\frac{1}{R} = \beta \geq e^{h_{top}(f)}$.

CONJECTURE:

Assume that for some $f \in \mathcal{CPM}$ the Key Equation is solvable for some $\lambda_0 > 1$. Then it is also solvable for each $\lambda > \lambda_0$!