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A class of cubic Rauzy Fractals

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Introduction

The Rauzy fractal is a compact subset of the space $\mathbb{R}^{d-1}$, $d \geq 2$. It has a fractal boundary and it induces two kind of tilings of $\mathbb{R}^{d-1}$, one of them is periodic and the other is auto-similar. Rauzy fractals are connected to many areas as substitution dynamical system, Number Theory among others (see [1],[2]). There are many ways of constructing Rauzy fractals, one of them is by β-expansions. Let $\beta > 1$ be a real number and $x \in \mathbb{R}^+$. Using greedy algorithm , we can write x in base β as $x = \sum_{i=-\infty}^{k} a_i \beta^i$ where $k \in \mathbb{Z}$ and a_i belong to the set A where $A = \{0, \ldots, \beta - 1\}$ if $\beta \in \mathbb{N}$ or $A = \{0, \ldots, \lfloor \beta \rfloor\}$ otherwise, where $\lfloor \beta \rfloor$ is the integer part of β.
The sequence $(a_i)_{i \leq k}$ is called β expansion of x and is also denoted by $a_k a_{k-1} \ldots$ The greedy algorithm can be defined as follows (see [3],[4]): denote by $\{y\}$ the fractional party of a number y. There exists an integer $k \in \mathbb{Z}$ such $\beta^k \leq x < \beta^{k+1}$. Let $x_k = \lfloor x/\beta^k \rfloor$ and $r_k = \{x/\beta^k\}$. Then for $i < k$, put $x_i = \lfloor \beta r_{i+1} \rfloor$ e $r_i = \{\beta r_{i+1}\}$. We get

$$x = x_k \beta^k + x_{k-1}\beta^{k-1} + \cdots$$

if $k < 0$ $(x < 1)$, we put $x_0 = x_{-1} = \cdots = x_{k+1} = 0$. If an expansion $(x_i)_{i \leq k}$ satisfies $x_i = 0$ for all $i < n$, it is said to be finite and the ending zeros are omitted. It will be denoted by $(x_i)_{n \leq i \leq k}$ or $x_k \ldots x_n$. For numbers $0 \leq x < 1$, the greedy expansion coincides with the β-representation of Rényi (see [10]) which can be defined by means of β-transformation of the interval $[0,1]$

$$T_\beta(x) = \{\beta x\},\ x \in [0,1].$$

For $x \in [0,1[$, we have $x_k = \lfloor \beta T_\beta^{k-1}(x) \rfloor$, but for $x = 1$, the two algorithms differ. The β-expansion of 1 is $1 = 1.0000\cdots$, while the Rényi β- representation of 1 is

$$d(1,\beta) = .t_{-1}t_{-2}\ldots,$$

where

$$t_{-k} = \lfloor \beta T_\beta^{k-1}(1) \rfloor,\ \forall k \geq 1.$$

We put

$$E_\beta = \{(x_i)_{i \geq k},\ k \in \mathbb{Z} \mid \forall n \geq k,\ (x_i)_{n \geq i \geq k} \text{ is a finite } \beta \text{ expansion } \}.$$

Now, assume that β is a Pisot number of degree $d \geq 3$, that means that β is an algebraic integer of degree d whose Galois conjugates have modulus less than one. We denote by $\beta_2, \ldots, \beta_r$ the real Galois conjugates of β and by

$$\beta_{r+1}, \ldots, \beta_{r+s}, \beta_{r+s+1} = \overline{\beta_{r+1}}, \ldots, \beta_{r+2s} = \overline{\beta_{r+s}}$$

its complex Galois conjugates. Let $\psi = (\beta_2, \ldots, \beta_{r+s}) \in \mathbb{R}^{r-1} \times \mathbb{C}^s$ and put $\psi^i = (\beta_2^i, \ldots, \beta_{r+s}^i)$ for all $i \in \mathbb{Z}$.
If $.x_{-1} \ldots x_{-N}$ is a finite β-expansion, we put

$$\mathcal{K}_{.x_{-1}\ldots x_{-N}} = \{\sum_{i=-N}^{+\infty} a_i \psi^i,\ (a_i)_{i \geq -N} \in E_\beta,\ a_i = x_i,\ \forall -N \leq i \leq -1\}.$$

The set $\mathcal{K}_{.x_{-1}\ldots x_{-N}}$ is a subset of $\mathbb{R}^{r-1} \times \mathbb{C}^s$ called a tile.
The Rauzy fractal is by definition the central tile

$$\mathcal{R} = \mathcal{K}_{.0} = \{\sum_{i=0}^{+\infty} a_i \psi^i,\ (a_i)_{i \geq 0} \in E_\beta\}.$$

It is a compact subset of $\mathbb{R}^{r-1} \times \mathbb{C}^s \approx \mathbb{R}^{d-1}$, and it induces a periodic tiling of the above space. That is there exists a group H which is isomorphic to $\mathbb{Z}^{d-1}$ such that $\mathbb{R}^{r-1} \times \mathbb{C}^s = \bigcup_{h \in H}(\mathcal{R} + h)$, moreover the intersection of $\mathcal{R}$ with the interior of another tile $\mathcal{R} + h$, is not empty. An important class of Pisot numbers are those such that the associated Rauzy fractal has 0 as an interior point. This numbers are characterized by Akiyama in [7]. They are exactly the Pisot numbers that satisfy

$$\mathbb{Z}[\beta] \cap [0, +\infty[\subset \text{Fin}(\beta) \text{ (called property (F))}\ ,$$

where Fin(β) is the set of nonnegative real numbers which have a finite β-expansion. On the other hand, Akiyama [5] characterized the set cubic unit Pisot's numbers that satisfy property (F) to be exactly the set of dominant roots of the following polynomial (with integer coefficients):

$$x^3 - ax^2 - bx - 1,\ a \geq 0 \text{ and } -1 \leq b \leq a+1.$$

(if $b = -1$ add the restriction $a \geq 2$).
In particular, this set divided into three subsets:

a) $0 \geq b \geq a$, and in this case $d(1,\beta) = \cdot ab1$,

b) $b = -1,\ a \geq 2$. In this case $d(1,\beta) = \cdot(a-1)(a-1)01$,

c) $b = a+1$, and in this case $d(1,\beta) = \cdot(a+1)00a1$.

The objective of this work is to study properties of the classical Rauzy fractal

$$\mathcal{R} = \mathcal{R}_a = \left\{\sum_{i=2}^{+\infty} a_i\alpha^i, a_i \in \{0,1,...,a-1\}, a_i a_{i-1} a_{i-2} a_{i-3} < (a-1)(a-1)01\right\}$$

and the G-Rauzy fractal (or Rauzy fractal with initial conditions)

$$\mathcal{G} = \mathcal{G}_a = \left\{\begin{array}{l} \sum_{i=2}^{+\infty} a_i\alpha^i, a_i \in \{0,1,...,a-1\}, a_2 < a, a_3a_2 < (a-1)(a-1) \\ a_4a_3a_2 < (a-1)(a-1)0, a_ia_{i-1}a_{i-2}a_{i-3} < (a-1)(a-1)01 \end{array}\right\}$$

in the case where $b = -1$ and $a \geq 2$.

1. Results

Here we present the results about the sets defined earlier.

Theorem 1 *The set $\mathcal{G}$ induces a periodic tiling of the complex plane, that is,*

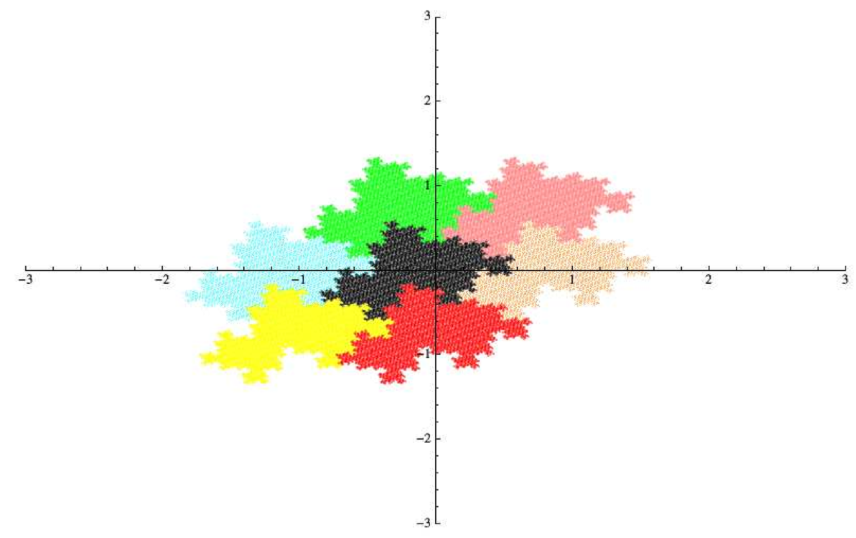
a) $\mathbb{C} = \bigcup_{u \in \mathbb{Z} + \mathbb{Z}\alpha}(\mathcal{G} + u)$;

b) $int(\mathcal{G} + u) \cap (\mathcal{G} + v) \neq \emptyset,\ u,\ v \in \mathbb{Z} + \mathbb{Z}\alpha$ *implies que* $u = v$.

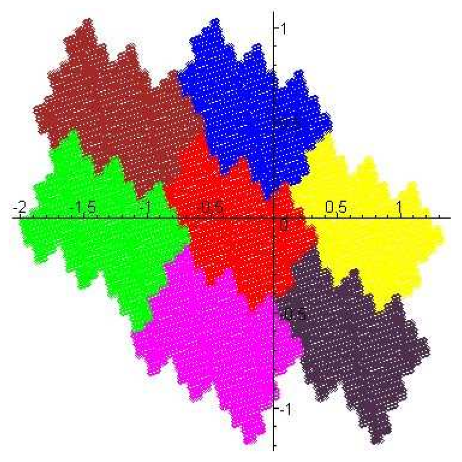
Theorem 2 *For all $a \geq 2$ we have*

$$\partial\mathcal{G}_a = \bigcup_{u \in B} \mathcal{G}_a \cap (\mathcal{G}_a + u)$$

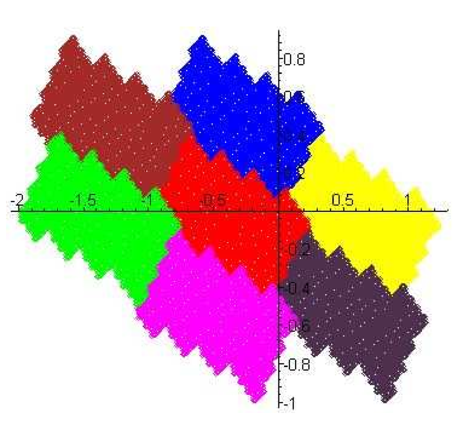
where $B = \{\pm 1, \pm\alpha, \pm(\alpha - 1)\}$.



$\mathcal{G}_2$



$\mathcal{G}_3$



$\mathcal{G}_4$

Theorem 3 *The set $\mathcal{R}$ induces a periodic tiling of the complex plane, that is,*

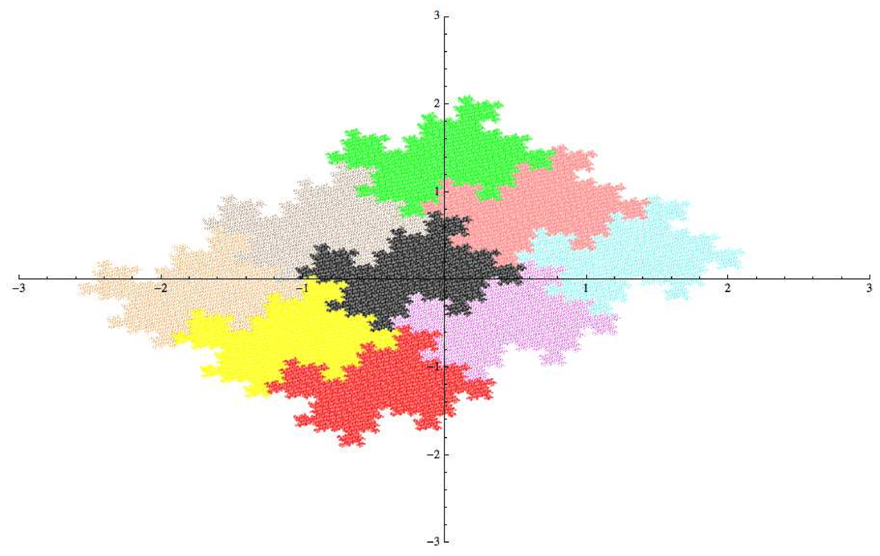
a) $\mathbb{C} = \bigcup_{u \in \mathbb{Z} + \mathbb{Z}\alpha}(\mathcal{R} + u)$;

b) $int(\mathcal{R} + u) \cap (\mathcal{R} + v) \neq \emptyset,\ u,\ v \in \mathbb{Z} + \mathbb{Z}\alpha$ *implies que* $u = v$.

Theorem 4 *For all $a \geq 2$ we have*

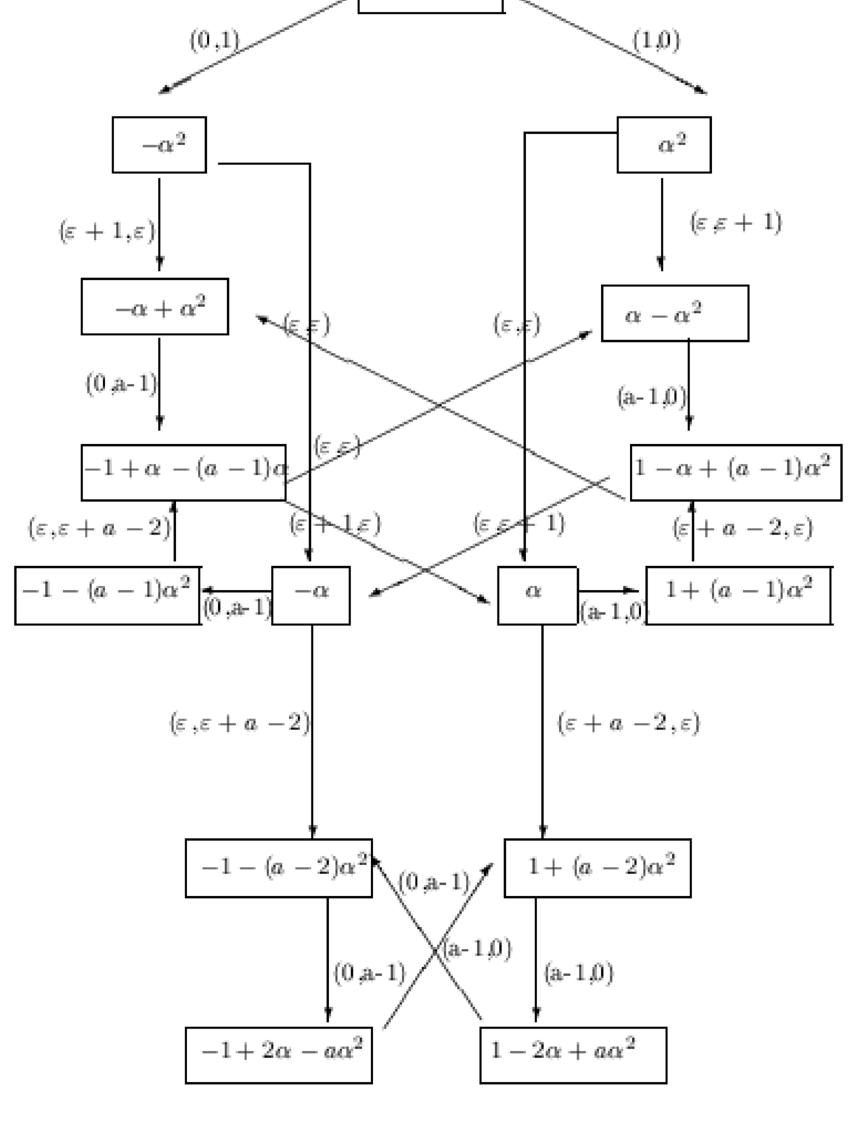
$$\partial\mathcal{R}_a = \bigcup_{x \in B} \mathcal{R}_a \cap (\mathcal{R}_a + x)$$

where $B = \{\pm u, \pm\alpha u, \pm(1+\alpha)u, \pm(1-\alpha)u\}, u = \alpha - 1.$



$\mathcal{R}_2$

In order to describe the fractal boundary of the sets $\mathcal{G}_a$ and $\mathcal{R}_a$ we use a finite automata (finite directed graph with states and arrows connecting these states) whose set of states are $S = \{0, \pm\alpha, \pm\alpha^2, \pm(\alpha - \alpha^2),\ \pm(1+(a-1)\alpha^2), \pm(1+(a-2)\alpha^2),\ \pm(1-\alpha+(a-1)\alpha^2),\ \pm(1-2\alpha+a\alpha^2)\}$ .We have that $\sum_{i=l}^{\infty} a_i\alpha^i = \sum_{i=l}^{\infty} b_i\alpha^i$ if only if, the sequence $((a_i,b_i))_{i \geq l}$ is an infinite path in automata starting from the initial state. The automata is given above



2. The case $a = 2$

Using the automata we can prove that each of the six regions $\mathcal{G} \cap (\mathcal{G} + u)$ which forms the boundary $\mathcal{G}_a$ is the image of one of them by affine functions. We can also show that there exists 3 affine functions $g_0,\ g_1,\ g_2$ such that

$$\mathcal{G}_{\alpha-1} = \bigcup_{i=0,1,2} g_i(\mathcal{G}_{\alpha-1}).$$

As consequence we can explicitly calculate the Hausdorff's dimension of $\partial\mathcal{G}$.

Theorem 5 - *$\mathcal{G}_{\alpha-1}$ satisfies the following properties:*

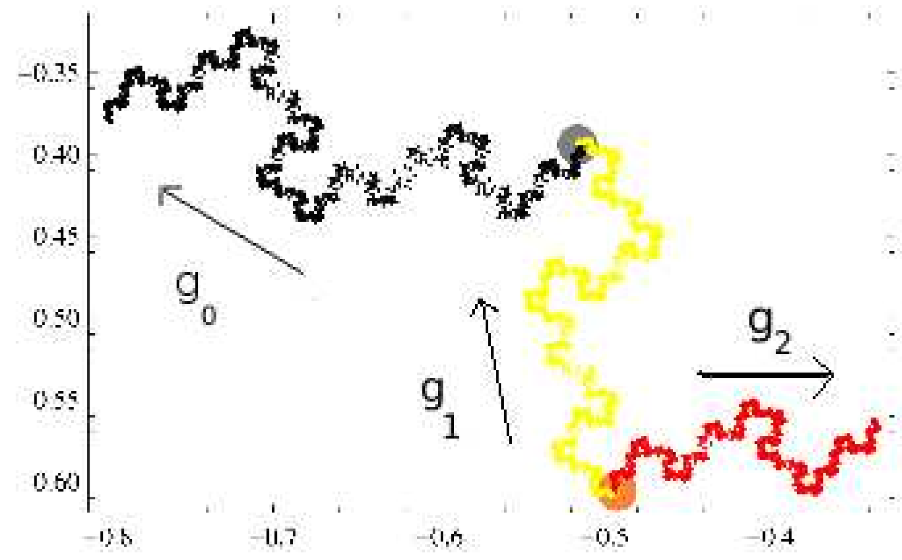
1. $\mathcal{G}_{\alpha-1} = g_0(\mathcal{G}_{\alpha-1}) \cup g_1(\mathcal{G}_{\alpha-1}) \cup g_2(\mathcal{G}_{\alpha-1})$;

2. $g_0(\mathcal{G}_{\alpha-1}) \cap g_1(\mathcal{G}_{\alpha-1}) = \{-(\alpha^{-1} + \alpha + \alpha^5)\}$;

3. $g_1(\mathcal{G}_{\alpha-1}) \cap g_2(\mathcal{G}_{\alpha-1}) = \{-(\alpha^{-1} + \alpha + \alpha^4 + \alpha^6)\}$;

4. $g_0(\mathcal{G}_{\alpha-1}) \cap g_2(\mathcal{G}_{\alpha-1}) = \emptyset$;

where $g_0(z) = \alpha^{-3} + \alpha^{-1} + \alpha^2 z,\ g_1(z) = -1 - \alpha^3 + \alpha^3 z$ *and* $g_2(z) = \alpha^2 + \alpha^5 + \alpha^4 z$.



Theorem 6 *There is a bijective continuous function*

$$f : [0,1] \longrightarrow \mathcal{G}_{\alpha-1}$$

such that $f(0) = -1 - \alpha^3$ *and* $f(1) = \alpha^2 - \alpha^3$.

Theorem 7 *1.* $\mathcal{G}_1 = -\alpha + \alpha^{-1}\mathcal{G}_{\alpha-1}$;

2. $\mathcal{G}_\alpha = -\alpha^2 + \mathcal{G}_{\alpha-1}$;

3. $\mathcal{G}_{1-\alpha} = -\alpha + 1 + \mathcal{G}_{\alpha-1}$;

4. $\mathcal{G}_{-\alpha} = -\alpha - \alpha^2 + \mathcal{G}_{\alpha-1}$;

5. $\mathcal{G}_{-1} = -1 - \alpha + \alpha^{-1}\mathcal{G}_{\alpha-1}$

We can use the results above to calculate the Hausdorff's dimension of $\mathcal{G}_{\alpha-1}$.

Theorem 8 - *Let A be a compact set of $\mathbb{C}$ such that*

$$A = \cup_{i=0}^{n}\varphi_i(A).$$

Suppose that $|\ \varphi_i(x) - \varphi_i(y)\ | = r_i\ |\ x - y\ |,\ \forall x,\ y \in \mathbb{C}$. *Then* $dim_H(A) \leq s$, *where s is the only real number which verifies* $\sum_{i=0}^{n} r_i^s = 1$.

When $\varphi_i(A)$ intersect in points it is known that $dim_H(A) = s$. Using Theorem [5], $g_0(\mathcal{R}_{\alpha-1}) \cap g_1(\mathcal{R}_{\alpha-1})$ and $g_1(\mathcal{R}_{\alpha-1}) \cap g_2(\mathcal{R}_{\alpha-1})$ are points and $g_0(\mathcal{R}_{\alpha-1}) \cap g_2(\mathcal{R}_{\alpha-1})$ is empty. Consequently $dim_H(\mathcal{R}_{\alpha-1}) = s$, where s verifies

$$|\ \alpha\ |^{2s} + |\ \alpha\ |^{3s} + |\ \alpha\ |^{4s} = 1.$$

Here we have that

$$dim_H(\mathcal{R}_{\alpha-1}) = \frac{\log\rho}{\log|\ \alpha\ |} = 1.359337357,$$

where ρ is the maximum root of polynomial $X^4 + X^3 + X^2 - 1 = 0$.

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