Limit cycles of a generalized Liénard differential equation via averaging theory

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Introduction

One of the main problems in the theory of ordinary differential equations is the study of their limit cycles, their existence, their number and their stability. A limit cycle of a differential equation is a periodic orbit in the set of all isolated periodic orbits of the differential equation. These last years hundreds of papers studied the limit cycles of planar polynomial differential systems. The Second part of the 16th Hilbert's problem is related with the least upper bound on the number of limit cycles of polynomial vector fields having a fixed degree. The generalized polynomial Liénard differential equation

$$\ddot{x} + f(x)\dot{x} + g(x) = 0. \tag{1}$$

was introduced in [?]. Here the dot denotes differentiation with respect to the time t, and $f(x)$ and $g(x)$ are polynomials in the variable x of degrees n and m respectively.

In this work, we study the maximum number of limit cycles of the Liénard polynomial differential system

$$\begin{cases} \dot{x} = y, \dot{y} = -x - \sum_{k\geq 1} \epsilon^k f_k(x,y)y. \end{cases} \tag{2}$$

which bifurcate from the periodic orbits of the linear differential system (2) with $\epsilon = 0$, using the averaging theory of first and second order.

1. Main Results

The main results of this paper are the following.

Theorem 1.
By applying the first order averaging method to the Liénard polynomial differential system (2) at most $[\frac{n}{2}]$ limit cycles bifurcate from the periodic orbits of the linear system (2) with $\epsilon = 0$.

Theorem 2.
By applying the second order averaging method to the Liénard polynomial differential system (2) at most $max\{[n + \frac{(-1)^{n+1}}{2}], [\frac{m}{2}]\}$ limit cycles bifurcate from the periodic orbits of the linear system (2) with $\epsilon = 0$.

2. The Averaging Theory of First and Second Orders

The averaging theory of first and second order for studying periodic orbits was developed in [1] and [2]. It is summarized as follows.

Consider the differential system

$$x'(t) = \epsilon F_1(t,x) + \epsilon^2 F_2(t,x) + \epsilon^3 R(t,x,\epsilon), \tag{3}$$

where $F_1, F_2 : R \times D \to R^n, R : R \times D \times (-\epsilon_f, \epsilon_f) \to R^n$ are continuous functions, T-periodic in the first variable, and D is an open subset of R^n. Assume that the following hypotheses (i) and (ii) hold.

(i) $F_1(t,.) \in C^1(D)$ for all $t \in R$, F_1, F_2, R, D_xF_1 are locally Lipschitz with respect to x, and R is differentiable with respect to ϵ. We define

$$F_{10}(z) = \frac{1}{T}\int_0^T F_1(s,z)ds,$$

$$F_{20}(z) = \frac{1}{T}\int_0^T \left[D_zF_1(s,z).y_1(s,z) + F_2(s,z)\right] ds,$$

where

$$y_1(s,z) = \int_0^s F_1(t,z)dt.$$

(ii) For $V \subset D$ an open and bounded set and for each $\epsilon \in (-\epsilon_f, \epsilon_f)$ $\{0\}$, there exists $a_\epsilon \in V$ such that $F_{10}(a_\epsilon) + \epsilon F_{20}(a_\epsilon) = 0$ and $d_B(F_{10} + \epsilon F_{20}, V, a_\epsilon) \neq 0$.

Then, for $|\epsilon| > 0$ sufficiently small there exists a T-periodic solution $\varphi(.,\epsilon)$ of the system 1.2 such that $\varphi(0,\epsilon) = a_\epsilon$.

The expression $d_B(F_{10} + \epsilon F_{20}, V, a_\epsilon) \neq 0$ means that the Brouwer degree of the function $F_{10} + \epsilon F_{20} : V \to R^n$ at the fixed point a_ϵ is not zero. A sufficient condition for the inequality to be true is that the Jacobian of the function $F_{10} + \epsilon F_{20}$ at a_ϵ is not zero.

If F_{10} is not identically zero, then the zeros of $F_{10} + \epsilon F_{20}$ are mainly the zeros of F_{10} for ϵ sufficiently small. In this case the previous result provides the *averaging theory of first order*.

If F_{10} is identically zero and F_{20} is not identically zero, then the zeros of $F_{10} + \epsilon F_{20}$ are mainly the zeros of F_{20} for ϵ sufficiently small. In this case the previous result provides the *averaging theory of second order*.
For more information about the averaging theory see [4] and [5].

2. Proof of Theorem 1

We shall need the first order averaging theory to prove Theorem 1.

In order to apply the first order averaging method we write system (2) with $k = 1$, in polar coordinates (r, θ) where $x = rcos(\theta)$, $y = rsin(\theta), r > 0$ In this way system (2) is written in the standard form for applying the averaging theory.

If we write $f_1(x,y) = \sum_{i+j=0}^n a_{ij}x^iy^j$ then system (2) becomes

$\dot{r} = -\epsilon \sum_{i+j=0}^n a_{ij}r^{i+j+1}cos^i(\theta)sin^{j+2}(\theta)$,

$\dot{\theta} = -1 - \epsilon \sum_{i+j=0}^n a_{ij}r^{i+j}cos^{i+1}(\theta)sin^{j+1}(\theta)$.

Now taking θ as the new independent variable, we obtain

$\frac{dr}{d\theta} = \epsilon \left(\sum_{i+j=0}^n a_{ij}r^{i+j+1}cos^i(\theta)sin^{j+2}(\theta)\right) + O(\epsilon^2)$

and

$F_{10}(r) = \frac{1}{2\pi}\int_0^{2\pi} \left(\sum_{i+j=0}^n a_{ij}r^{i+j+1}cos^i(\theta)sin^{j+2}(\theta)\right) d\theta.$

In order to calculate the exact expression of F_{10} we use the following formula

$\int_0^{2\pi} cos^i(\theta)sin^{j+2}(\theta)d\theta = 0 \quad if \quad i \; odd \quad or \quad j \; odd$

$\int_0^{2\pi} cos^i(\theta)sin^{j+2}(\theta)d\theta = \alpha_{ij} \quad if \quad i \; even \quad and \quad j \; even,$
Hence

$$F_{10}(r) = \frac{1}{2\pi}\sum_{i+j=0}^{n} a_{ij}\alpha_{ij}r^{i+j+1} \quad where \; i \; even \; and \; j \; even. \tag{4}$$

Then the polynomial $F_{10}(r)$ has at most $[\frac{n}{2}]$ positive roots, and we can choose the coefficients a_{ij} with i even and j even in such a way that $F_{10}(r)$ has exactly $[\frac{n}{2}]$ simple positive roots. Hence Theorem 1 is proved.

3. Proof of Theorem 2

For proving Theorem 2, we shall use the second order averaging theory. If we write $f_1(x,y) = f(x,y) = \sum_{i+j=0}^n a_{ij}x^iy^j$ and $f_2(x,y) = g(x,y) = \sum_{i+j=0}^m b_{ij}x^iy^j$ then system (2) with $k = 2$ in polar coordinates $(r,\theta), r > 0$ becomes

$\dot{r} = -\epsilon\left(\sum_{i+j=0}^n a_{ij}r^{i+j+1}cos^i(\theta)sin^{j+2}(\theta)\right) - \epsilon^2\left(\sum_{i+j=0}^m b_{ij}r^{i+j+1}cos^i(\theta)sin^{j+2}(\theta)\right)$,

$\dot{\theta} = -1 - \epsilon r\left(\sum_{i+j=0}^n a_{ij}r^{i+j+1}cos^{i+1}(\theta)sin^{j+1}(\theta)\right) - \epsilon^2 r\left(\sum_{i+j=0}^m b_{ij}r^{i+j+1}cos^{i+1}(\theta)sin^{j+1}(\theta)\right)$.

Taking θ as the new independent variable, we obtain

$\frac{dr}{d\theta} = \epsilon F_1(\theta,r) + \epsilon^2 F_2(\theta,r) + O(\epsilon^3)$,

Where $F_1(\theta,r) = \sum_{i+j=0}^n a_{ij}r^{i+j+1}cos^i(\theta)sin^{j+2}(\theta)$,
$F_2(\theta,r) = \sum_{i+j=0}^m b_{ij}r^{i+j+1}cos^i(\theta)sin^{j+2}(\theta) - r\cos(\theta)\sin(\theta)\left[\sum_{i+j=0}^n a_{ij}r^{i+j}cos^i(\theta)sin^{j+1}(\theta)\right]^2$.

Now we determine the corresponding function F_{20}. For this we put $F_{10} \equiv 0$ which is equivalent to $a_{ij} = 0$ for all i even and j even, and we compute

$\frac{d}{dr}F_1(\theta,r) = \sum_{i+j=0}^n (i+j+1)a_{ij}r^{i+j}cos^i(\theta)sin^{j+2}(\theta)$,

$y_1 = \int_0^\theta F_1(\phi,r)d\phi$ which is equal to

$a_{10}r^2\left(\alpha_{110}sin(\theta) + \alpha_{210}sin(3\theta)\right) + ... + a_{lb}r^{l+b+1}$
$\left(\alpha_{1lb}sin(\theta) + \alpha_{2lb}sin(3\theta) + ... + \alpha_{\frac{(l+b+2)+1}{2}lb}sin((l+b+2)\theta)\right)$
$+a_{01}r^2\left(\alpha_{101} + \alpha_{201}\cos(\theta) + \alpha_{301}\cos(3\theta)\right) + ... + a_{cd}r^{c+d+1}$
$\left(\alpha_{1cd} + \alpha_{2cd}\cos(\theta) + \alpha_{3cd}\cos(3\theta) + ... + \alpha_{\frac{(c+d+2)+3}{2}cd}\cos((c+d+2)\theta)\right)$
$+a_{11}r^3\left(\alpha_{111} + \alpha_{211}\cos(2\theta) + \alpha_{311}\cos(4\theta)\right) + ... + a_{ld}r^{l+d+1}$
$\left(\alpha_{1ld} + \alpha_{2ld}\cos(2\theta) + \alpha_{3ld}\cos(4\theta) + ... + \alpha_{\frac{(l+d+2)+2}{2}ld}\cos((l+d+2)\theta)\right)$.

Where l is the greatest odd number and b is the greatest even number so that $l+b$ is less than or equal to n. c is the greatest even number and d is the greatest odd number so that $c+d$ is less than or equal to n. α_{ijk} are real constants exhibited during the computation of $\int_0^\theta cos^i(\phi)sin^{j+2}(\phi)d\phi$ for all i and j. We know from (4) that F_{10} is identically zero if and only if $a_{ij} = 0$ for all i even and j even.

Moreover
$\int_0^{2\pi} cos^i(\theta)sin^{j+2}(\theta)sin((2k+1)\theta)d\theta = 0 \quad if \quad i \; odd \; and \; \forall j \in N,$
$= A_{ij}^{2k+1} \neq 0 \quad if \quad i \; even \quad and \quad j \; odd, \; k = 0,1,...$
$\int_0^{2\pi} cos^i(\theta)sin^{j+2}(\theta)d\theta \neq 0, \quad if \, and \, only \, if \quad i \; even \quad and \quad j \; even,$
$\int_0^{2\pi} cos^i(\theta)sin^{j+2}(\theta)cos((2k+1)\theta)d\theta = 0 \quad if \; j \; odd \quad and \; \forall i \in N,$
$= B_{ij}^{2k+1} \neq 0 \quad if \quad i \; odd \quad and \quad j \; even, \; k = 0,1,...$
$\int_0^{2\pi} cos^i(\theta)sin^{j+2}(\theta)cos((2k)\theta)d\theta = 0, \quad \forall i,j \in N \; with \; i \; odd \; or \; j \; odd, \; k = 0,1,...$

So

$\int_0^{2\pi} ddrF_1(\theta,r)y_1(\theta,r)d\theta =$

$\sum_{i+j=1}^n (i+j+1)a_{ij}r^{i+j}\int_0^{2\pi} cos^i(\theta)sin^{j+2}(\theta)\times$

$[a_{10}r^2(\alpha_{110}\sin(\theta) + \alpha_{210}\sin(3\theta)) + ... + a_{lb}r^{l+b+1}(\alpha_{1lb}\sin(\theta) + \alpha_{2lb}\sin(3\theta) + ... + \alpha_{(l+b+2)+12lb}\sin((l+b+2)\theta))]$

$+\sum_{i+j=1}^n (i+j+1)a_{ij}r^{i+j}\int_0^{2\pi} cos^i(\theta)sin^{j+2}(\theta)\times$

$[a_{01}r^2(\alpha_{101} + \alpha_{201}\cos(\theta) + \alpha_{301}\cos(3\theta)) + ...+$

$a_{cd}r^{c+d+1}(\alpha_{1cd} + \alpha_{2cd}\cos(\theta) + \alpha_{3cd}\cos(3\theta) + ... + \alpha_{(c+d+2)+32cd}\cos((c+d+2)\theta))]$.

Then

$\int_0^{2\pi} ddrF_1(\theta,r)y_1(\theta,r)d\theta = \sum_{i+j=1}^n (i+j+1)a_{ij}r^{i+j}$

$\times[a_{10}r^2(\alpha_{110}A_{ij}^1 + \alpha_{210}A_{ij}^3) + ... + a_{lb}r^{l+b+1}(\alpha_{1lb}A_{ij}^1 + \alpha_{2lb}A_{ij}^3 + ... + \alpha_{(l+b+2)+12lb}A_{ij}^{l+b+2})]$

$+\sum_{i+j=1}^n (i+j+1)a_{ij}r^{i+j}$

$\times[a_{01}r^2(\alpha_{201}B_{ij}^1 + \alpha_{301}B_{ij}^3) + ... + a_{cd}r^{c+d+1}(\alpha_{2cd}B_{ij}^1 + \alpha_{3cd}B_{ij}^3 + ... + \alpha_{(c+d+2)+32cd}B_{ij}^{c+d+2})]$.

Moreover

$\int_0^{2\pi} F_2(\theta,r)d\theta = \sum_{i+j=0}^m b_{ij}r^{i+j+1}\int_0^{2\pi} cos^i(\theta)\sin^{j+2}(\theta)d\theta$

$-2\sum_{i_{odd}+j_{even}=1}^n \; \sum_{k_{even}+h_{odd}=1}^n \; a_{ij}\; a_{kh}\; r^{i+j+k+h+1}$

$$\int_0^{2\pi} cos^{i+k+1}(\theta)\sin^{j+h+3}(\theta)d\theta.$$

but

$\int_0^{2\pi} cos^i(\theta)\sin^{j+2}(\theta)d\theta = 0 \quad if \quad i \; odd \; or \; j \; odd,$
$= C_{ij} \neq 0 \quad if \quad i \; even \; and \; j \; even,$

and $\int_0^{2\pi} cos^{i+k+1}(\theta)\sin^{j+h+3}(\theta)d\theta = C_{(i+k+1)(j+h+1)} \neq 0.$

Hence

$\int_0^{2\pi} F_2(\theta,r)d\theta = b_{00}C_{00}r + (b_{02}C_{02} + b_{20}C_{20})r^3 + ... + (\sum_{i+j=m}^{i\; even, j\; even} b_{ij}C_{ij})r^{m+1} - a_{10}a_{01}C_{22}r^3 - ... - 2\sum_{i_{odd}+j_{even}=n}\sum_{k_{even}+h_{odd}=n} a_{ij}\; a_{kh}\; C_{(i+k+1)(j+h+1)}r^{i+j+k+h+1}$

Then $F_{20}(r)$ is the polynomial

$r\sum_{i+j=1}^n (i+j+1)a_{ij}r^{i+j}$
$\times[a_{10}r(\alpha_{110}A_{ij}^1 + \alpha_{210}A_{ij}^3) + ... + a_{lb}r^{l+b}(\alpha_{1lb}A_{ij}^1 + \alpha_{2lb}A_{ij}^3 + ... + \alpha_{(l+b+2)+12lb}A_{ij}^{l+b+2})$
$+ a_{01}r(\alpha_{201}B_{ij}^1 + \alpha_{301}B_{ij}^3) + ... + a_{cd}r^{c+d}(\alpha_{2cd}B_{ij}^1 + \alpha_{3cd}B_{ij}^3 + ... + \alpha_{(c+d+2)+32cd}B_{ij}^{c+d+2})$
$+ b_{00}C_{00} + (b_{02}C_{02} + b_{20}C_{20})r^2 + ... + (\sum_{i+j=m}^{i\; even, j\; even} b_{ij}C_{ij})r^m$
$- a_{10}a_{01}C_{22}r^2 - ... - 2\sum_{i_{odd}+j_{even}=n}$
$\sum_{k_{even}+h_{odd}=n} a_{ij}\; a_{kh}\; C_{(i+k+1)(j+h+1)}r^{i+j+k+h}]$.

We conclude that:

1. if $m > 2n$, F_{20} has at most $[m2]$ positive roots.
2. if $m \leq 2n$ we have
   - **(a)** if n is odd: $l+b = n$, $c+d = n$ and $i+j+k+h = 2n$. Then, F_{20} has at most n positive roots.
   - **(b)** if n is even: $l+b = n-1$, $c+d = n-1$ and $i+j+k+h = 2n-2$. Then, F_{20} has at most $n-1$ positive roots.

Then
F_{20} has at most $max\{[n + (-1)^{n+1}2], [m2]\}$ positive roots. This completes the proof of Theorem 2.

References

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