Chaotic behavior of the solution C_0-semigroup of the von Foerster-Lasota equation in different phase spaces.

Javier Aroza[1] and Elisabetta Mangino[2]

[1] Institut Universitari de Matemàtica Pura i Aplicada, Universitat Politècnica de València, València, Spain.
[2] Dipartimento di Matematica e Fisica "E. De Giorgi", Università del Salento, Lecce, Italy

Introduction

The usually so-called von Foerster-Lasota equation, or McKendrick equation, is one of the first age-dependent models in dynamics of population. We will see that the solution C_0-semigroup of this equation is chaotic on $L^p[0,1]$ and in the subspace of the Sobolev spaces $W^{1,p}(0,1)$ of functions vanishing at the origin, where $p \in [1,+\infty)$. The approach is different from that previously used by [2], since it is based on an accurate estimate of the behavior of the eigenvectors of the generator of the semigroup. The previous results have been generalized to the study of first order linear parabolic equation with a more general drift (for more details we refer the reader to [1]).

From now on, let X be an infinite-dimensional separable complex Banach space.

- A one-parameter family $\mathcal{T} = \{T_t : X \to X \ ; \ t \geq 0\}$ of continuous linear maps is a *C_0-semigroup* if the following three conditions are satisfied:
  - $T_0 = I$,
  - $T_t T_s = T_{t+s}$, for all $t, s \geq 0$
  - $\lim_{t \to s} T_t x = T_s x$, for each $x \in X$ and $s \geq 0$.
- The *orbit* of a point $x \in X$ is the set $\mathcal{O}(x, \mathcal{T}) = \{T_t x : t \geq 0\}$.
- A C_0-semigroup $\mathcal{T}$ is called *hypercyclic* if there is an $x \in X$ whose orbit under $\mathcal{T}$ is dense in X.
- We define the set of *periodic points* of $\mathcal{T}$ as follow:
  $Per(\mathcal{T}) := \{x \in X \ : \ T_t x = x \ for \ some \ t > 0\}$.
- $\mathcal{T}$ is *chaotic* if $\mathcal{T}$ is hypercyclic and $Per(\mathcal{T})$ is dense in X.
- $\mathcal{T}$ is said to be *mixing* if, for any pair of nonempty open sets $U, V \subset X$, there exists some $t_0 \geq 0$ such that $T_t(U) \cap V \neq \emptyset$, for all $t \geq t_0$.
- The C_0-semigroup $\mathcal{T}$ has an *infinitesimal generator*, denoted by $(A, D(A))$, where A is defined on a dense subspace $D(A) \subset X$ by the formula $Ax = \lim_{t \to 0}(T_t x - x)/t$, $x \in D(A)$.

We will use a criterion related with the eigenvalues of the C_0-semigroup, the so-called **Godefroy-Shapiro criterion**:

Theorem 1: *Let T be an operator defined on a separable complex Banach space X. If the following subspaces are dense in X:*

$$Y_1 := span\{x \in X \ ; \ Tx = \mu x \ for \ some \ \mu \in \mathbb{C} \ with \ |\mu| < 1\}$$

$$Y_2 := span\{x \in X \ ; \ Tx = \mu x \ for \ some \ \mu \in \mathbb{C} \ with \ |\mu| > 1\}$$

Then T is mixing, and in particular hypercyclic. In addition, if also

$$Y_3 := span\{x \in X \ ; \ Tx = e^{\theta \pi i} x \ for \ some \ \theta \in \mathbb{Q}\}$$

is dense in X, then T is chaotic.

If we denote by $C_c^1 = C_c^1(I)$ the space of continuously differentiable functions with compact support in $I = (a,b)$, we can define the Sobolev space $W^{1,p} = W^{1,p}(I)$ as follow:

$$W^{1,p} := \{u \in L^p; \ \exists g \in L^p \ : \ \int_a^b u\varphi' = -\int_a^b g\varphi, \ \forall \varphi \in C_c^1\}.$$

The Sobolev space is a separable Banach space equipped with the norm:

$$\|u\|_{W^{1,p}} = \|u\|_{L^p} + \|u'\|_{L^p}.$$

In this case u' denote the derivative in the sense of the distributions but it is well known that if $u \in C^1 \cap L^p$ and $u' \in L^p$ (where now u' denote the usually derivative) then $u \in W^{1,p}$. Moreover it is known that $W^{1,p}$ embeds continuously into the space of continuous functions on $[a,b]$.

We are especially interested in an intermediate Sobolev space, we denote this space as $W_*^{1,p} = W_*^{1,p}(I)$ defined by:

$$W_*^{1,p} := \{v \in W^{1,p}; \ v(a) = 0\}$$

This space equipped with the norm of $W^{1,p}$ is a separable Banach space for $p \in [1, +\infty[$.

Proposition: (**Poincaré inequality**)
Let I be a bounded interval. Then there exists a real constant C, depending on I, such that

$$\|u\|_{W^{1,p}} \leq C\|u'\|_{L^p}, \ \textit{for all } u \in W_*^{1,p}.$$

Then we can use the equivalent norm $\|u\|_{W_*^{1,p}} = \|u'\|_{L^p}$.

Proposition: (**Dual representation**)
Let I be an open bounded interval. Let $F \in (W_*^{1,p})'$ and $1/p + 1/q = 1$. Then there exists $f \in L^q$ such that

$$\langle F, u \rangle = \int_I f u', \ \textit{for all } u \in W_*^{1,p}$$

and $\|F\|_{(W_*^{1,p})'} = \|f\|_{L^q}$.

Section 1

In this section we present the first results about von Foerster-Lasota equation. First of all fix $I = (0,1)$ and $p \in [1, +\infty)$, suppose that $\lambda > 0$ and $v : [0,1] \to \mathbb{R}$. The von Foerster-Lasota equation is:

$$\frac{\partial u}{\partial t} + x\frac{\partial u}{\partial x} = \lambda u, \qquad t \geq 0, \ 0 \leq x \leq 1,$$
$$u(0,x) = v(x), \qquad 0 \leq x \leq 1.$$

The solution of this equation is defined by the following C_0-semigroup on $W_*^{1,p}$ and on L^p:

$$(T_t^\lambda v)(x) = u(t,x) = e^{\lambda t} v(xe^{-t}), \quad x \in [0,1], \ t \geq 0. \qquad (1)$$

Consider the real values $-\infty < a < \lambda - 1 + (1/p)$ and $b \in \mathbb{R}$.
We can define the following family of functions continuous and differentiable on $(0,1]$:

$$v_{a,b}(x) := \begin{cases} x^{\lambda - a}\cos(-b \ln x), & \text{if } x > 0, \\ 0, & \text{if } x = 0. \end{cases}$$

$$w_{a,b}(x) := \begin{cases} x^{\lambda - a}\sin(-b \ln x), & \text{if } x > 0, \\ 0, & \text{if } x = 0. \end{cases}$$

Proposition: (**Eigenfunctions and periodic points**)
The vectors $v_{a,b}(x), w_{a,b}(x) \in W_*^{1,p}$ for every $a < \lambda - 1 + (1/p)$ and $b \in \mathbb{R}$. Moreover, if $u_{a,b}(x) = v_{a,b}(x) + i w_{a,b}(x)$ we have that $T_t^\lambda u_{a,b} = e^{(a+ib)t} u_{a,b}$ and $T_t^\lambda u_{0,(2n\pi)/t} = u_{0,(2n\pi)/t}$, for every $n \in \mathbb{N}$.

In order to use the Godefroy-Shapiro criterion for the proof of the main results we need discuss the modulus of the eigenvalues $\mu = e^{(a+ib)t}$, i.e., $|\mu| = e^{at}$, then the different values to consider are:

$$|\mu| < 1 \longleftrightarrow a < 0,$$
$$|\mu| > 1 \longleftrightarrow a > 0,$$
$$|\mu| = 1 \longleftrightarrow a = 0 \text{ or } t = 0.$$

Theorem 2:
Suppose that $\lambda > 1 - 1/p$. Then the C_0-semigroup given by the equation (1), $\{T_t^\lambda\}_{t \geq 0}$, associated with the von Foerster-Lasota equation is mixing and chaotic on $W_*^{1,p}$.

Theorem 3:
Consider the problem of von Foerster-Lasota equation with the C_0-semigroup $\{T_t^\lambda\}_{t \geq 0}$ given by the equation (1). If $\lambda > -1/p$, then the C_0-semigroup is mixing and chaotic on L^p.

Section 2

In this section we consider the case of what happens if we replace the parameter λ in von Foerster-Lasota equation by a continuous function $\gamma : [0,1] \to \mathbb{R}$. The equation is defined by:

$$\frac{\partial u}{\partial t} + x\frac{\partial u}{\partial x} = \gamma(x) u, \qquad t \geq 0, \ 0 \leq x \leq 1. \qquad (2)$$

There is no loss of generality in assuming that we change the assumptions in this study by $\gamma(0) = \lambda$, to prove this, it only remains to find conditions for the map $\gamma(x)$. This idea goes back at least as far as [3].

The essential relationship between the above case and this generalization is the expression of the solution C_0-semigroup. If we consider the function

$$g(x) = -\int_x^1 \frac{\gamma(s) - \gamma(0)}{s} ds, \qquad x \in (0,1],$$

the solution C_0-semigroup on $W_*^{1,p}$ and on L^p of the equation (2) is the following:

$$(T_t^\gamma u)(x) = e^{\gamma(0)t} e^{g(x)} e^{-g(xe^{-t})} u(xe^{-t}), \quad x \in [0,1], \ t \geq 0.$$

Now, just the fact that these C_0-semigroups associate to the operators T_t^γ and $T_t^{\gamma(0)}$ are quasiconjugate lets us prove that the equation (2) have properties as hypercyclicity, mixing and chaos, because this properties are preserved under quasiconjugacy.

Lemma:
Let K be a positive constant and let $M_m : W_*^{1,p} \to W_*^{1,p}$ be an operator defined by $M_m f = mf$, where $m(x) = e^{-g(x)}$. If

$$\left|\frac{\gamma(x) - \gamma(0)}{x}\right| \leq K$$

then the operator M_m is continuous and bijective and the following diagram commute:

$$\begin{tikzcd} W_*^{1,p} \arrow[r, "T_t^\gamma"] \arrow[d, "M_m"'] & W_*^{1,p} \arrow[d, "M_m"] \\ W_*^{1,p} \arrow[r, "T_t^{\gamma(0)}"'] & W_*^{1,p} \end{tikzcd}$$

Hence we have the immediate result:

Theorem 4:
Suppose that $\gamma(0) > 1 - 1/p$ and the assumptions of the previous Lemma. Then the C_0-semigroup $\{T_t^\gamma\}_{t \geq 0}$, associated with the generalized von Foerster-Lasota equation (2), is mixing and chaotic on $W_*^{1,p}$.

To conclude, we can be extend this result for L^p with another conditions for the function $\gamma(x)$.

Theorem 5:
Assume that $\gamma(0) > -1/p$ and

$$\frac{\gamma(x) - \gamma(0)}{x} \in L^1(0,1).$$

Then the C_0-semigroup $\{T_t^\gamma\}_{t \geq 0}$ is mixing and chaotic on L^p.

Section 3

In this section we mention a generalization of Section 1. In this case we consider the following equation:

$$\frac{\partial u}{\partial t} = F(x)\frac{\partial u}{\partial x} + \lambda u, \qquad t \geq 0, \ 0 \leq x \leq 1.$$

We assume that F is a locally Lipschitz function such that the solution $\phi(\cdot, x_0)$ of the inital value problem $x' = F(x)$ exists for every $t \geq 0$ and $x_0 \in [0,1]$.
Then the solution semigroups of this equation is given by $T_t f(x) = f(\phi(t,x))$, $t \geq 0$, $x \in [0,1]$. If for every $t \geq 0$ $\phi(t, \cdot)$ is continuously differentiable and there exists $M > 0$, $\omega \in \mathbb{R}$ such that $|\partial_x \phi(t,x)| \geq Me^{\omega t}$ for every $t \geq 0$, $x \in [0,1]$, then it is known that the semigroup $\{T_t\}_{t \geq 0}$ is strongly continuous semigroup in L^p (see [6]).

Theorem 6:
Under the previous assumptions, let $v_a(x) = e^{(\lambda - a)\int_x^1 \frac{ds}{F(s)}}$, where $\lambda > 0$, $a < \lambda$. If $v_a \in L^p$ for every $a < \lambda$, then the C_0-semigroup $\{T_t\}_{t \geq 0}$ is mixing and chaotic on L^p.

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UNIVERSITAT POLITÈCNICA DE VALÈNCIA

IUMPA Instituto Universitario de Matemática Pura y Aplicada

Dipartimento di Matematica e Fisica "Ennio De Giorgi"

UNIVERSITÀ DEL SALENTO