Behavior of the binary collision in a planar restricted $(N+1)$-body problem

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Outline

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The planar $(N+1)$–body problem consists in the study of the motion of an infinitesimal mass which is subject to the gravitational attraction of N other larger masses (called *primaries*) which move on a central configuration solution. While the primaries determine the dynamics of the infinitesimal mass m_{N+1}, their motion is not affected by the presence of m_{N+1}.

A configuration $\mathbf{q} = (\mathbf{q}_1, \cdots, \mathbf{q}_N) \in \mathbb{R}^{2N}$ is a central configuration for the masses $m_1, \ldots, m_N$ if there exists a constant $\lambda \in \mathbb{R}$ such that

$$\lambda\, \mathbf{q}_j = \sum_{l \neq j} m_j \frac{\mathbf{q}_l - \mathbf{q}_j}{\|\mathbf{q}_l - \mathbf{q}_j\|^3}.$$

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Equation of motions in an inertial systems

We denote by $\mathbf{r}$ the position of the infinitesimal mass on the xy plane and by $\mathbf{r}_j(t) = e^{i\omega t}\mathbf{r}_j^0 \quad j = 1, 2, \cdots, N$, the position of the primaries which move on the plane with angular velocity ω and $\mathbf{r}_j^0$ form a central configuration of the N-body problem.

According to the Newton laws the motion of the massless particle is governed by the second differential equations

$$\ddot{\mathbf{r}} = \nabla V(\mathbf{r}, t),$$

where the Newtonian potential is given by

$$V(\mathbf{r}, m_1, \cdots, m_N; t) = \sum_{j=1}^{N} \frac{m_j}{\|\mathbf{r} - \mathbf{r}_j(t)\|},$$

and m_j are the masses of the primaries.

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Relative equilibrium and central configurations

Associated to each central configuration $(\mathbf{q}_1, \cdots, \mathbf{q}_N)$ we can define a particular solution of the N-body problem as

$$\mathbf{r}_j(t) = e^{i\omega t}\mathbf{q}_j, \quad j = 1, \cdots, N, \tag{1}$$

where the particles rotate with the same angular velocity ω. Since the equations of motion of the N-body problem are given by

$$\ddot{\mathbf{r}}_j = \sum_{\substack{l=1 \\ j \neq l}}^{N} m_l \frac{\mathbf{r}_l - \mathbf{r}_j}{\|\mathbf{r}_l - \mathbf{r}_j\|^3}, \tag{2}$$

we must have $\omega^2 = -\lambda$.

Restriction

Using the fact that a rotation and a dilation of a central configuration is still a central configuration, we will assume without loss of generality that $\omega = 1$, i.e., $\lambda = -1$ and the body of mass m_1 is located in $(a, 0)$ with $a > 0$.

Equations of motion in rotating coordinates

In the planar $(N+1)$-body problem the equations of motion of the massless particle m_{N+1} in synodical coordinates (x, y) are given by the system of differential equations

$$\begin{aligned} \ddot{x} - 2\dot{y} &= \Omega_x, \\ \ddot{y} + 2\dot{x} &= \Omega_y \end{aligned} \tag{3}$$

where

$$\Omega = \Omega(x, y) = \frac{1}{2}(x^2 + y^2) + W(x, y), \tag{4}$$

and

$$W = W(x, y) = \sum_{j=1}^{N} \frac{m_j}{\rho_j}, \tag{5}$$

with $\rho_j = \sqrt{(x - x_j)^2 + (y - y_j)^2}$, $\mathbf{q}_1 = (a, 0)$ and $\mathbf{q_j} = (x_j, y_j)$ for $j = 2, \cdots, N$.

First integral

The system has the Jacobian first integral $C = C(x, y, X, Y)$ given by

$$C = 2\Omega(x, y) - (\dot{x}^2 + \dot{y}^2).$$

In order to write the Hamiltonian formulation of the $(N+1)$–body problem we introduce the new variables

$$\begin{aligned} x &= x, & y &= y, \\ X &= \dot{x} - y, & Y &= \dot{y} + x. \end{aligned} \tag{6}$$

Hence, it is verified that system (3) is equivalent to an autonomous Hamiltonian system with two degrees of freedom with Hamiltonian function given by

$$H = H(x, y, X, Y) = \frac{1}{2}(X^2 + Y^2) + yX - xY - W(x, y), \tag{7}$$

and the associated Hamiltonian system becomes

$$\begin{aligned} \dot{x} &= y + X, & \dot{X} &= Y + W_x, \\ \dot{y} &= -x + Y, & \dot{Y} &= -X + W_y. \end{aligned} \tag{8}$$

Regions of allowable motion for m_{N+1}

The set

$$\mathcal{R}_C = \{(x,y) \in \mathbb{R}^2 \mid 2\Omega(x,y) \geq C\}$$

corresponds to the so called Hill region.

$$\Omega = \frac{1}{2}(x^2+y^2) + \sum_{j=1}^{N} \frac{m_j}{\sqrt{(x-x_j)^2+(y-y_j)^2}},$$

Remarks

1. $x \to \infty$ or $-\infty$ (respectively, $y \to \infty$ or $-\infty$) on the $\partial\mathcal{R}_C$ ($2\Omega = C$), then $C \to \infty$.
2. C very large implies that m_{N+1}can be sufficiently close to one of the primaries ($(x,y) \to (x_j, y_j)$), or escape.

If C is large enough then the infinitesimal particle can collide or can escape.

According to the Hill region, for sufficiently large value of C there are N bounded regions in the Hill regions and we shall suppose that the infinitesimal mass body (m_{N+1}) rotates around the primary of mass m_1.

The Hamiltonian close the binary collision

Firstly, we begin by considering the change to canonical coordinates given by

$$\begin{aligned}(\xi, \eta) &= (x - a, y) \\ (p_\xi, p_\eta) &= (X, Y - a).\end{aligned}$$

Thus, the Hamiltonian function (7) can be written as

$$H = \frac{1}{2}\left(p_\xi^2 + p_\eta^2\right) + \eta p_\xi - \xi p_\eta - a\xi - \frac{a^2}{2} - U(\xi, \eta) \qquad (9)$$

where

$$U(\xi, \eta) = \frac{m_1}{\sqrt{\xi^2 + \eta^2}} + \sum_{j=2}^{N} \frac{m_j}{\sqrt{(\xi - (x_j + a))^2 + (\eta - y_j)^2}}.$$

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The position and momentum coordinates of m_{N+1} will be denoted, respectively, by the complex numbers

$$z = \xi + i\eta, \qquad \mathcal{Z} = p_\xi + ip_\eta.$$

The Hamiltonian (9) that describes the motion is then written as

$$H(z, \mathcal{Z}) = \frac{1}{2}|\mathcal{Z}|^2 + \frac{1}{2}i(\bar{z}\mathcal{Z} - z\bar{\mathcal{Z}}) - a\mathrm{Re}(z) - \frac{a^2}{2} - U(z) \quad (10)$$

where

$$U(z) = \frac{m_1}{|z|} + \sum_{j=2}^{N} \frac{m_j}{|z - w_j|}$$

with $w_j = x_j + a + iy_j$, $j = 2, \cdots, N$ and $\mathrm{Re}(z)$ is the real part of the complex number z.

Let us define $\tilde{H} = 2H + a^2$, and we obtain the following Hamiltonian system

$$\frac{dz}{dt} = \frac{\partial \tilde{H}}{\partial \bar{\mathcal{Z}}}, \qquad \frac{d\mathcal{Z}}{dt} = -\frac{\partial \tilde{H}}{\partial \bar{z}} \tag{11}$$

with

$$\tilde{H}(z, \mathcal{Z}) = |\mathcal{Z}|^2 + i(\bar{z}\mathcal{Z} - z\bar{\mathcal{Z}}) - 2a\mathrm{Re}(z) - \frac{2m_1}{|z|} - 2\tilde{U}(z), \tag{12}$$

where

$$\tilde{U}(z) = \sum_{j=2}^{N} \frac{m_j}{|z - w_j|}. \tag{13}$$

We expand the function $\tilde{U}(z)$ defined in (13) in Taylor series yielding

$$\tilde{U}(z) = A_1 + a\mathrm{Re}(z) + \frac{1}{2}(A_2 z^2 + \bar{A}_2 \bar{z}^2) + A_4 |z|^2 + O_3(z, \bar{z})$$

Parameters associated the central configuration

$$
\begin{aligned}
\tilde{U}(0,0) &= \sum_{j=2}^{N} m_j \frac{1}{|w_j|} := A_1, \\
\frac{\partial \tilde{U}}{\partial z}(0,0) &= \frac{1}{2}\sum_{j=2}^{N} m_j \frac{\bar{w}_j}{|w_j|^3} = \frac{a}{2}, \\
\frac{\partial \tilde{U}}{\partial \bar{z}}(0,0) &= \frac{1}{2}\sum_{j=2}^{N} m_j \frac{w_j}{|w_j|^3} = -\frac{a}{2}, \\
\frac{\partial^2 \tilde{U}}{\partial z^2}(0,0) &= \frac{3}{4}\sum_{j=2}^{N} m_j \frac{\bar{w}_j^2}{|w_j|^5} := A_2, \\
\frac{\partial^2 \tilde{U}}{\partial \bar{z}^2}(0,0) &= \frac{3}{4}\sum_{j=2}^{N} m_j \frac{w_j^2}{|w_j|^5} := A_3 = \bar{A}_2, \\
\frac{\partial^2 \tilde{U}}{\partial z \partial \bar{z}}(0,0) &= \frac{1}{4}\sum_{j=2}^{N} m_j \frac{1}{|w_j|^3} := A_4.
\end{aligned}
$$

We replace these changes in the Hamiltonian $\tilde{H}$, and we end up with:

$$\tilde{H}(z, \mathcal{Z}) = |\mathcal{Z}|^2 + i(\bar{z}\mathcal{Z} - z\bar{\mathcal{Z}}) - \frac{2m_1}{|z|} - (A_2 z^2 + \bar{A}_2 \bar{z}^2) - 2A_4|z|^2 + O_3(z, \bar{z}).$$

Remark

The system has a singularity at $z = 0$, which corresponds to a collision between m_1 and m_{N+1}.

The singularity is regularized by a transformation due to Levi–Civita as follows: we define a small parameter ε as $C = 1/\varepsilon^2$, in such a way that we restrict our attention to the surface energy $\tilde{H} + 1/\varepsilon^2 = 0$, and introduce new variables (z,w) by a canonical transformation with a scaling of time

$$z = 2x^2, \qquad \mathcal{Z} = \frac{w}{\varepsilon \bar{x}}, \qquad \frac{dt}{dt'} = 2\varepsilon |z|$$

Remark

Noted that is a double covering $\mathbb{C}^* \times \mathbb{C} \to \mathbb{C}^* \times \mathbb{C}$ identifying the pairs (x, w) and $(-x, -w)$, where $\mathbb{C}^* = \mathbb{C} \setminus \{0\}$.

Regularization

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Regularization

The resulting Hamiltonian is

$$K(x, w) = \varepsilon^2 |x|^2 \left[\tilde{H}\left(2x^2, \frac{w}{\varepsilon \bar{x}}\right) + \frac{1}{\varepsilon^2} \right] \tag{14}$$

and the corresponding vector field is

$$\frac{dx}{dt'} = \frac{\partial K}{\partial \bar{w}} = w + \cdots, \qquad \frac{dw}{dt'} = -\frac{\partial K}{\partial \bar{x}} = -z + \cdots \tag{15}$$

which has an equilibrium point at $x = w = 0$.

We have the Hamiltonian regularized

$$\begin{aligned} K(x,w) = & \quad |w|^2 + [1 + 2i\varepsilon(\bar{x}w - x\bar{w})]|x|^2 - m_1\varepsilon^2 - \varepsilon^2 g(x) \\ = & \quad |w|^2 + |x|^2 + 2i\varepsilon(\bar{x}w - x\bar{w})|x|^2 - m_1\varepsilon^2 - \varepsilon^2 g(x) \end{aligned} \tag{16}$$

where

$$g(x) = |x|^2 \left[4(A_2 x^4 + \bar{A}_2 \bar{x}^4) + 8A_4|x|^4 + O_6(x,\bar{x})\right]. \tag{17}$$

Remark

From (16) that if one does not consider the last term $\varepsilon^2 g(x)$, we obtain the *Kepler problem in rotating coordinates*, or the *rotating Kepler problem.*

Hamiltonian regularized

Notice that equation (16) can be rewritten as

$$K(x,w) = \begin{array}{l} |w|^2 + |x|^2 + 2i\varepsilon(\bar{x}w - x\bar{w})|x|^2 - m_1\varepsilon^2 - \\ \varepsilon^2[4(A_2x^4 + \bar{A}_2\bar{x}^4)\,|x|^2 + 8A_4|x|^6 + O_8(x,\bar{x})], \end{array} \tag{18}$$

$K = 0$, truncating at a certain order

- at order 2, the harmonic oscillator, which regularizes the Kepler problem;
- at order 4, the regularization of the Kepler problem in a rotating frame;
- at order 6, the generalized Hill's problem. In fact, if $A_2 = \overline{A}_2 = \frac{3\mu}{4}$ and $A_4 = \frac{\mu}{4}$, $\mu \in (0, 1/2]$ we have the Hill problem.

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- We restrict our attention to the energy surface $H + 1/\varepsilon^2 = 0$, i.e., $K = 0$. Then the solutions of the new equations correspond to those of the original problem.
- It is shown that the system derived from K has an equilibrium position at $x = w = 0$ and that if C is large enough (i.e., ε is sufficiently small), the integral surface $K = 0$ is close to this equilibrium point.
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Remarks

Conley, C. On some new long periodic solutions of the plane restricted three body problem. *Comm. Pure Appl. Math.* **16** (1963), 449-467.

He proved the existence of the periodic solutions of long period for the planar restricted circular three body problem.

In fact, he showed there exist a negative constant E_0 independent of the ratio of the masses such that in each energy $H = E < E_0$, there exists a periodic solution with arbitrarily long period.

Remarks

Chenciner, A.; Llibre, J. A note on the existence of invariant punctured tori in the planar circular restricted three-body problem. *Ergodic Theory Dynam. Systems* **8*** (1988), Charles Conley Memorial Issue, 63-72.

- If $\mu\nu \neq 0$ and ε is small enough, the intersection of the "circle" $x = 0$ with its image P_ε consist of exactly eight transversal points.
- If $\mu\nu \neq 0$, in any neighbourhood of $\varepsilon = 0$ in $\mathbb{R}^+$, there exists an interval of values of ε such that the "circle" $x = 0$ intersects an uncountable number of invariant curves of P_ε, each in a finite number points. Moreover, in each of these intervals there are at least two values of ε such that the circle "x=0" contains a pair of points α, $P_\varepsilon(\alpha)$ belonging to the same invariant curve of P_ε.

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Our objective

Our goal in this work is to give a generalization of the Conley thesis results. In addition , we show that the Hill terms (the terms of sixth order) are of the same nature but with different coefficients, which allow us to give the differences with respect to known results. Thus we point out conditions on the relative equilibrium of the N–body problem in order to overcame the difficulties.

In particular, we want to adapt the previous techniques in order to get a characterization of the flow in a neighborhood of one binary collision in the restricted $(N+1)$–body problem under some conditions of the potential W and the fixed central configuration.

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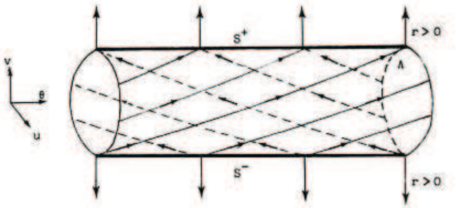
Remark

This work grew out of an attempt to carry over the methods of the study of the restricted three body problem for high values of the Jacobian constant by Conley, Chenciner and Chenciner-Llibre applying their techniques to a more general restricted problem.

The flow of the Kepler problem

The Kepler problem (i.e., $\varepsilon = 0$) in rotating variables has been studied in McGehee coordinates (r, v, θ, u). From these studies we know:

- there are two circles of equilibrium points, S^- and S^+;
- the set of collision (respectively ejection) orbits on the Jacobian level $C = c$ corresponds to a cylinder W_c^s (respectively W_c^u) which is simultaneously the unstable manifold associated to S^- and the stable manifold associated to S^+.



For each $m_1 > 0$ and $C = c \in \mathbb{R}$ (fixed) taking the sets W_c^s and W_c^u

- In particular, W_c^s is the set of orbits which ends at collision (resp., in W_c^u beginning at collisions) and forms a cylinder in a neighborhood of the collision manifold.
- We are interested in the intersection of these invariant manifolds at the equilibrium point.
- These intersections will give orbits which tend to the binary collision between m_1 and m_{N+1} when $t \to \pm\infty$.

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Transversality: The perturbed Kepler problem

We denote by γ_c^u (resp. γ_c^s) the first intersection of the invariant manifold W_c^u in forward (resp. W_c^s in backward) time with $v = 0$ in the Jacobian level $C = c$.

Theorem

Assume that $m_1 m_2 \cdots m_N \neq 0$, $A_2 \neq 0$, $C = 1/\varepsilon^2$ *with* ε *small enough, then* γ_c^s *and* γ_c^u *are homeomorphic to a circle, and intersect transversally in four points. In particular there are four ejection–collision orbits in our restricted* $(N+1)$*–body problem.*

Remark

In particular this theorem tell us that there are four ejection-collision orbits in our restricted $(N+1)$–body problem.

Proof

The proof is bases in the ideas of Sanders (1982) about the Melnikov method. Also during the proof is necessary to assume that the relative equilibrium must satisfies $A_2 \neq 0$.

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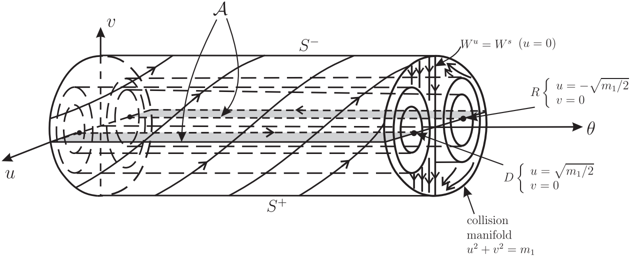
Blow-up

Blow-up the singularity due to the binary collision $x = 0$ of the bodies m_1 and m_{N+1}, by the standard McGehee-like coordinates

$$x = \varepsilon r e^{i\theta}, \qquad v + iu = \frac{rw}{x}, \qquad \frac{dt'}{ds} = r.$$

We prove that for small ε there is an annulus $\mathscr{A}$ defined on integral surface which is a surface of section.

The construction of a bicontinuous mapping P_ε from the annulus onto itself is carried out, and the conditions of the Poincaré–Birkhoff fixed-point theorem are verified. The fixed points of the iterates of P_ε correspond to long–periodic solutions of the differential equations of motion.



In order to prove the existence of periodic and almost–periodic solutions close to a binary collision we need to construct the Poincaré map up to the sixth order in ε because we will use the Twist Moser Theorem.

In polar coordinate $(\varphi,\ \rho)$, the Poincaré map is given by

$$P_\varepsilon(\varphi,\rho) = \left(\varphi + \frac{1}{2} - \frac{m_1}{2}\varepsilon^3 - \frac{3}{2}m_1(1 - m_1 A_4)\varepsilon^6\rho + O(\varepsilon^7), \rho + O(\varepsilon^7)\right)$$

Consequences

Theorem

If $m_1 m_2 \cdots m_N \neq 0$, $A_2 \neq 0$, $1 - A_4 > 0$ *and* ε *is small enough, the restricted* $(N+1)$*-body problem possesses an infinite number of periodic solutions (in the rotating frame) of long periods and also possesses an infinite number of almost–periodic solutions.*

The Twist Moser Theorem

Theorem (Invariante Curve)

Let $0 < \gamma \leq 1$, $\mathcal{C} > 0$, $\beta \geq 0$ *be three real numbers,* ω *a real number satisfying* $\forall\ p/q$, $|\omega - p/q| \geq \gamma\mathcal{C}/|q|^{2+\beta}$, *and* F *a real analytic embedding of* $(\mathbb{R}/\mathbb{Z}) \times [-\frac{1}{4}, \frac{1}{4}]$ *into* $(\mathbb{R}/\mathbb{Z}) \times \mathbb{R}$,

$$F(\phi, \tau) = (\phi + \omega + \gamma\tau + \gamma\Phi_1(\phi, \tau), \sigma + \gamma\Phi_2(\phi, \tau)).$$

Suppose that F *has the "intersection property"(i.e. any embedded closed curve going around the annulus must intersect its image under* F*) and consider a neighborhood*
$\mathscr{A} = \{(\phi, \tau)$ *such that* $|\mathrm{Im}\,\varphi| \leq a$, $\tau \in \mathscr{A}'\}$ *of* $(\mathbb{R}/\mathbb{Z}) \times [-\frac{1}{4}, \frac{1}{4}]$ *in* $(\mathbb{C}/\mathbb{Z}) \times \mathbb{C}$ *on which the complex extension of* F *is defined. For each* $\eta > 0$ *there is a* $\delta > 0$ *depending on* $\mathcal{C}$, β, $\mathscr{A}$ *but not on* γ, *such that, if the* C^0 *norms on* $\mathscr{A}$ *of* Φ_1 *and* Φ_2 *satisfy* $\|\Phi_1\|_{\mathscr{A}} + \|\Phi_2\|_{\mathscr{A}} < \delta$, *there exists a unique real analytic function* $\psi : \mathbb{R}/\mathbb{Z} \to [-\frac{1}{4}, \frac{1}{4}]$ *whose graph is an invariant curve of* F *on which* F *is analytically conjugate to the rotation* $\phi \to \phi + \omega$ *and such that* $\|\psi\|_0 < \eta$ *(* $\|\cdot\|_0$ *is the* C^0 *norm).*

Applications: The Restricted circular three body problem

We start with the circular restricted three body problem. In this case the potential is given by

$$W(x,y) = \frac{\nu}{\sqrt{(x-\mu)^2+y^2}} + \frac{\mu}{\sqrt{(x+\nu)^2+y^2}}, \tag{19}$$

with $\mu+\nu=1$. In this case we have for $\mu \in (0,1/2)$:

1. $a=1$ and the two primaries are in the same circle.
2. $W(x,-y)=W(x,y)$.
3. $m_1=\nu$.
4. $A_2=\overline{A}_2=\frac{3}{4}\mu \neq 0$.
5. $A_4=\frac{1}{4}\mu$.

Applications: The equilateral restricted four body problem

The potential in this case is given by

$$W = W(x,y) = \frac{1-2\mu}{\rho_1} + \frac{\mu}{\rho_2} + \frac{\mu}{\rho_3}, \tag{20}$$

and $\rho_1 = \sqrt{(x-\sqrt{3}\mu)^2 + y^2}$,
$\rho_2 = \sqrt{(x+\frac{\sqrt{3}}{2}(1-2\mu))^2 + (y-\frac{1}{2})^2}$,
$\rho_3 = \sqrt{(x+\frac{\sqrt{3}}{2}(1-2\mu))^2 + (y+\frac{1}{2})^2}$.
In this case we have for $\mu \in (0, 1/2)$:

1. $a = \sqrt{3}\mu$ and the three primaries are in the same circle.
2. $W(x,-y) = W(x,y)$.
3. $m_1 = 1-2\mu$.
4. $A_2 = \overline{A}_2 = \frac{3}{4}\mu \neq 0$.
5. $A_4 = \frac{1}{4}\mu$.

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Jaume Llibre

happy 60th birthday

60 anys

Fa vint anys que dic que fa vint anys que tinc vint anys i encara tinc força, i no tinc l'ànima morta i em sento bullir la sang ...

Joan Manuel Serrat