On generation of independent quadratic conserved quantities

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Abstract. In a Hamiltonian system one can produce a conserved quantity from two conserved quantities by using the Poisson bracket. Jacobi considered this remark as the "deepest discovery of Poisson", while other authors, as Bertrand, remarked that nobody ever discovered a new conserved quantity by using this process.

(...)

Hans Lundmark observed a more spectacular way of producing new conserved quantities from two given ones. With his advisor Stefan Rauch-Wojciechowski, they considered another class of equations, that they call the Newton systems, where, in a vector space of dimension n, a force depends on the position and defines the second derivative of the position with respect to time. Then two conserved quantities that are quadratic in the velocities produce $n-2$ other ones. The theorem also works on a spherical space. In the Neumann problem on a n-dimensional sphere, starting with the energy and another quadratic conserved quantity, one produces in this way a (known) system of n quadratic independent conserved quantities in involution.

Recently, we found with Lundmark a simple criterion for the functional independence of conserved quantities produced in such a way. We present the result quite simply, using the "projective dynamics" point of view, i.e. the properties of central projection in dynamics discovered by Appell in 1890.

tentative statement

Tentative statement

Consider a system of the form $\ddot{q} = f(q)$, $q = (q_1, \dots, q_n) \in R^n$.
This is the motion of a particle under a field of forces f, or of several particles interacting, e.g. n-body problem.
A first integral is called quadratic if it is of degree 2 in the velocities $\dot{q}$. Note that the highest order term is always a polynomial in $(q, \dot{q})$, of degree at most 2 in q (and homogeneous of degree 2 in $\dot{q}$. Examples: energy, square of angular momentum, eccentricity vector of the 2-body problem, etc.).

"Theorem": If the system has two quadratic first integrals which are "sufficiently distinct", and which are "natural Hamiltonians" after a change of time and position variables of a certain kind, then there are indeed n independent quadratic first integrals and the system is Liouville-integrable.

typical example

The typical example

Let $a_0, a_1, \ldots, a_n$ be real parameters.

$$F_0 = \frac{1}{2}\sum_{i=1}^{n} \frac{\dot{q}_i^2}{a_0 - a_i} + \frac{1}{2(1 + q_1^2 + \cdots + q_n^2)}.$$

After $p_i = \dot{q}_i/(a_0 - a_i)$ this is a natural Hamiltonian. This is a quadratic first integral of Hamilton's equations which are of the form $\ddot{q} = f(q)$. The "other sufficiently distinct Hamiltonian" is

$$H = \frac{1}{2}((1 + q_1^2 + \cdots + q_n^2)(\dot{q}_1^2 + \cdots + \dot{q}_n^2) - (q_1\dot{q}_1 + \cdots)^2) +$$

$$+ \frac{a_0 + a_1 q_1^2 + \cdots + a_n q_n^2}{2(1 + q_1^2 + \cdots + q_n^2)}$$

statement again

The statement again

We will introduce a "dictionary" that allows us to express precisely the statement and to prove it. This dictionary may be called "projective dynamics" (theory of the central projection introduced by Appell, also named "gnomic projection" by Higgs). Using this dictionary the statement becomes purely algebraic.

Theorem (again). Consider two symmetric $n \times n$ matrices A and B. Expand $\bigwedge^{n-2}(A + \lambda B) = C_0 + \lambda C_1 + \cdots + \lambda^{n-2} C_{n-2}$. If $C_0, \ldots, C_{n-2}$ are linearly dependent then there is a λ such that $\bigwedge^{n-2}(A + \lambda B) = 0$.

We proved recently this with Hans Lundmark. Indeed we proved by induction the same statement for rectangular complex matrices, and for any exterior power.

dictionary

The dictionary

All these matrices correspond to the quadratic term of the quadratic first integrals. They are always first integrals of the "trivial" equation $\ddot{q} = 0$. Let us consider the degree 1 case before the degree 2. The linear first integrals of $\ddot{q} = 0$ are $\dot{q}_1$, $\dot{q}_2$, $\ldots$, $\dot{q}_n$, $q_1\dot{q}_2 - q_2\dot{q}_1$, $q_1\dot{q}_3 - q_3\dot{q}_1, \ldots$.
There are two lists of first integrals. The dictionary transforms these two lists into just one list. We add a variable q_0 and its derivative $\dot{q}_0$. We change any polynomial first integral

$$F(q_1, \ldots, q_n, \dot{q}_1, \ldots, \dot{q}_n)$$

into its "homogeneous form", function of $q_0, q_1, \ldots, q_n, \dot{q}_0, \ldots, \dot{q}_n$:

$$F(q_1/q_0, \ldots, q_n/q_0, q_0\dot{q}_1 - q_1\dot{q}_0, \ldots, q_0\dot{q}_n - q_n\dot{q}_0).$$

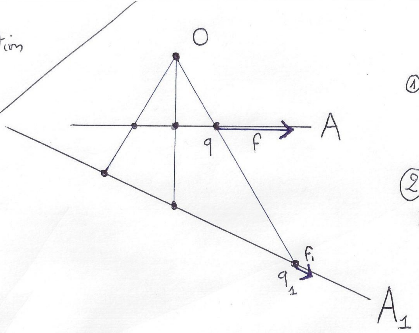
Then $\dot{q}_i$ becomes $q_0\dot{q}_i - q_i\dot{q}_0$ and the others are unchanged.

interpretation

Interpretation of $(q_0, \dot{q}_0)$

By adding this pair of variable we do not want to add one degree of freedom. We consider $q_0 = 1$, $\dot{q}_0 = 0$ as a restriction (it gives the old form from the homogeneous form) and we want to consider other similar restrictions. For example, taking the homogeneous form of H in the example above, then restricting H to the unit sphere $q_0^2 + \dots q_n^2 = 1$, we find the Hamiltonian of the classical Neumann problem on the sphere. But let us begin with simpler cases.

Th: Central projection sends trajectories of $\ddot{q} = f(q)$ to trajectories of $\ddot{q}_1 = f_1(q_1)$



Start with force f

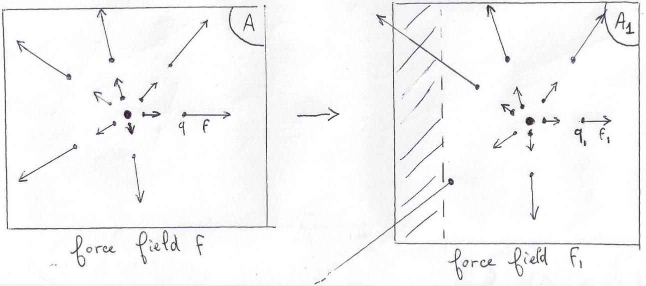
① PUSH FORWARD f by central projection

② Rescale

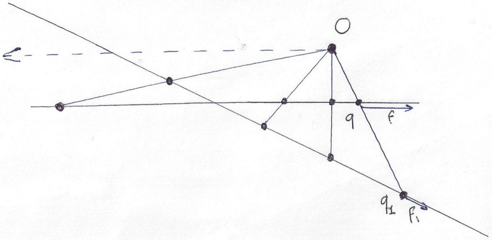
Rescaling factor $= \left(\frac{Oq}{Oq_1}\right)^4$

→ get f_1
New force at q_1

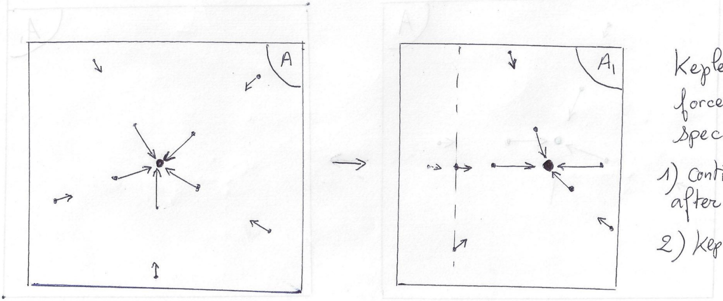
$\ddot{q} = f(q)$



$\ddot{q}_1 = f_1(q_1)$



Keplerian "Conic sections" becomes Keplerian Cones

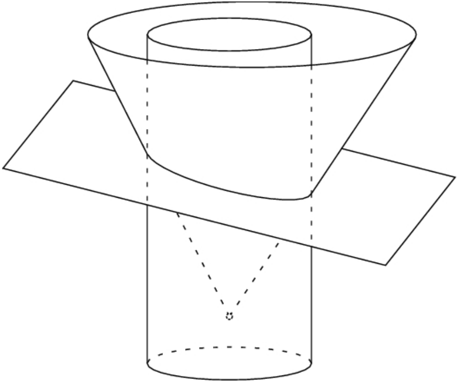


Kepler central force is special

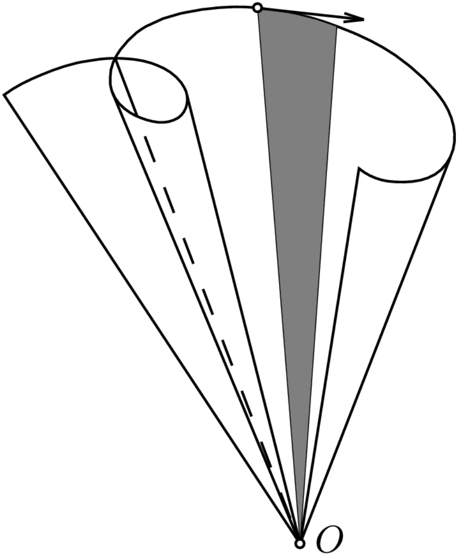
1) continuation after ∞

2) Kepler → Kepler

Keplerian semi-cone



Semi-cone



quadratic first integrals

Quadratic first integrals

As we saw, linear first integrals of $\ddot{q} = 0$ are linear forms in the variable $Q \wedge \dot{Q}$, where $Q = (q_0, q_1, \ldots, q_n)$. In the same way, quadratic first integrals of $\ddot{q} = 0$ are quadratic forms in the same variable. Quadratic forms on decomposable bivectors are well-known objects. We associate them to tensors R_{ijkl} such that $R_{ijkl} = -R_{jikl} = -R_{ijlk}$ and $R_{ijkl} + R_{jkil} + R_{kijl} = 0$. This implies that R defines a symmetric form on the bivectors, i.e. that $R_{ijkl} = R_{klij}$, but the cyclic (Bianchi) identity gives more, and the quadratic form is defined only by its values on decomposable bivectors. Such tensors have "Young tableau symmetry" with tableau

1	3
2	4

Special quadratic first integrals

$$E^* \quad \bigwedge^2 E^* \quad \bigwedge^3 E^* \quad \ldots \quad \bigwedge^{n-2} E^*$$

$$\downarrow \quad \downarrow\uparrow \quad \downarrow \quad \ldots \downarrow$$

$$E \quad \bigwedge^2 E \quad \bigwedge^3 E \quad \ldots \quad \bigwedge^{n-2} E$$

The key linearity

We will have a first integral with leading term R for the projective force field M if the form

$$R(q, M, q, .) \qquad (*)$$

is closed. We consider the case $R = g^{-1} \wedge g^{-1}$, set $g\xi_1 = q_1$, $g\xi_2 = q_2$, $g\xi = q$ and compute:

$$\partial_{q_1} R(q, M, q, q_2) = \partial_{q_1}\big(\langle \xi, q\rangle\langle \xi_2, M\rangle - \langle \xi_2, q\rangle\langle M, \xi\rangle\big) =$$

$$= 2\langle \xi_1, q\rangle\langle \xi_2, M\rangle + \langle \xi, q\rangle\langle \xi_2, \partial_{q_1} M\rangle - \langle \xi_2, q_1\rangle\langle M, \xi\rangle$$

$$-\langle \xi_2, q\rangle\langle \partial_{q_1} M, \xi\rangle - \langle \xi_2, q\rangle\langle M, \xi_1\rangle.$$

We subtract the terms exchanging 1 and 2 to get the expression of the differential of $(*)$. This is

$$3\langle \xi_1, q\rangle\langle M, \xi_2\rangle - 3\langle \xi_2, q\rangle\langle M, \xi_1\rangle + \langle \xi, q\rangle\big(\langle \xi_2, \partial_{q_1} M\rangle - \langle \xi_1, \partial_{q_2} M\rangle\big)$$

$$-\langle \xi_2, q\rangle\langle \partial_{q_1} M, \xi\rangle + \langle \xi_1, q\rangle\langle \partial_{q_2} M, \xi\rangle.$$

As M is homogeneous of degree -3, we have $\partial_q M = -3M$.

Let $\partial M \in V^* \otimes V$ be the differential of the vector field M at the point q. The contraction $g\partial M \in V \otimes V$ seems to be the good object. For example we will write Euler relation like this: $\xi g \partial M = -3M$. The formula above becomes

$$-\langle q \rfloor \xi_1 \wedge \xi_2, \xi g \partial M \rangle + \langle \xi, q \rangle \langle g \partial M, \xi_1 \wedge \xi_2 \rangle + \langle q \rfloor \xi_1 \wedge \xi_2, g \partial M \xi \rangle$$

$$= \langle g \partial M, (q \rfloor \xi_1 \wedge \xi_2) \wedge \xi + \langle \xi, q \rangle \xi_1 \wedge \xi_2 \rangle =$$

$$= \langle g \partial M, q \rfloor (\xi_1 \wedge \xi_2 \wedge \xi) \rangle = \langle \sigma | q \rfloor (\xi_1 \wedge \xi_2 \wedge \xi) \rangle,$$

where we set $\sigma = g\partial M - {}^t(g\partial M)$ and where the factor 2 is taken into account in the change of notation for the duality bracket, passing from general on tensors to particular on exterior algebra. The final equation is linear in g:

$$\xi \rfloor (q \wedge \sigma) = 0.$$

Then if g_1 and g_2 satisfy the relation, $g_1 + \lambda g_2$ also, then $\bigwedge^{n-2}(g_1 + \lambda g_2)$ defines a quadratic first integral for any λ and the coefficients of the expansion give quadratic first integrals.