Abelian integrals of general rational 1-forms are defined over $\mathbb{Q}$

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If a planar vector field with a polynomial Hamiltonian $H \in \mathbb{R}[x, y]$ is perturbed in a polynomial 1-parametric family

$$\dot{x} = \frac{\partial H}{\partial y} + \varepsilon Q(x, y), \qquad \dot{y} = -\frac{\partial H}{\partial x} - \varepsilon P(x, y),$$

then the limit cycles which bifurcate from a nonsingular oval on a level curve $\{H(x, y) = z\}$, $z \in \mathbb{R}$, correspond to the zeros of the Abelian integral

$$0 = \oint_{H=z} P(x, y)\,\mathrm{d}x + Q(x, y)\,\mathrm{d}y, \qquad P, Q \in \mathbb{R}[x, y]. \tag{1}$$

Establishing an explicit upper bound for the number of isolated zeros (roots) of the integral (1) in terms of the degrees of the polynomials H, P, Q was called the infinitesimal Hilbert 16th problem. While many low degree cases where well studied since late 1960-ies, when the problem was first formulated, the general double exponential bound was achieved only in 2010 [1].

However, the result of [1] fails to address the case where both the Hamiltonian and the perturbation form $\omega = P\mathrm{d}x + Q\mathrm{d}y$ are merely rational, not necessarily polynomial. While the proof of [1] can be relatively easily modified to cover the case of a rational Hamiltonian H, the appearance of poles for the form ω is considerably more difficult to overcome. One of the immediate reasons is that, while integrals of polynomial 1-forms of degree $\leqslant d$ form a finite-dimensional linear space, the integrals of rational 1-forms do not.

Dynamically, the rational perturbations naturally appear in the study of integrable polynomial vector fields with non-isolated singularities. Some of the simplest cases with the quadratic first integral where considered by J. Llibre in many publications with various co-authors (see, e.g., [3, 4]). They discovered quite a few peculiar properties of the integral (1) with a rational form ω as an analytic function of z. For instance, this function generically has ramification points of finite order, whereas a generic polynomial integral (1) has only logarithmic ramification points for finite values of z.

However, it turns out that even in the general rational case the integral (1) can be expressed via a suitable family of Q-functions, the class of transcendental functions defined by Pfaffian integrable systems with quasiunipotent monodromy over $\mathbb{Q}$ introduced in [2]. The key idea is to show that certain integrals satisfy a Picard–Fuchs-type system of equations with rational coefficients. Unlike the polynomial case where a simple algorithm of deriving such a system exists, in the rational case we only prove the existence of such a system along with an explicit upper bound for the complexity of its coefficients. This is sufficient to prove the double exponential upper bound for the number of isolated roots of a general Abelian integral, settling thus completely the infinitesimal Hilbert 16th problem for perturbations of foliations with algebraic first integrals.

References

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