Firing map for periodically and almost-periodically driven integrate-and-fire models: a dynamical systems approach

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We consider the so-called integrate-and-fire system, in which a continuous dynamics induced by the differential equation $\dot{x} = f(t, x)$, $f : \mathbb{R}^2 \to \mathbb{R}$, is interrupted by the threshold and reset behaviour: $\lim_{t \to s^+} x(t) = x_{\mathrm{r}}$ if $x(s) = x_{\mathrm{tr}}$. The question is to describe the sequence of consecutive resets s_n for a trajectory starting at time t_0 from the resting value x_{r} as iterations of some map $\Phi^n(t_0) = s_n$, called the *firing map*, and the sequence of *interspike-intervals* $\eta_n(t) = s_n - s_{n-1}$ as a sequence of displacements $\Phi^n(t_0) - \Phi^{n-1}(t_0)$ along a trajectory of this map. The problem appears in various applications, for example in modelling of an action potential (spiking) by a neuron ([1, 2, 3]).

We investigate behaviour of the sequence of interspike-intervals when the function $f(t, x)$ is smooth enough and periodic in t. In this case the problem is covered by analysis of the displacement sequence of an orientation preserving homeomorphism (diffeomorphism) φ of the circle ([4]), which is a projection of the firing map Φ onto S^1. If the firing rate, which is the rotation number $\varrho(\Phi)$, is rational, say $\varrho(\Phi) = p/q$, then $\eta_n(t)$ is asymptotically periodic with frequency q. If $\varrho(\Phi) \notin \mathbb{Q}$, then the values of $\eta_n(t)$ are dense in a set which depends on the map γ (semi-) conjugating the firing phase map φ with the rotation by ϱ and which is the support of the displacements distribution with respect to the invariant measure of φ. Further, with the use of topological dynamics, we discuss the recurrent properties of the sequence $\eta_n(t)$. We show how these results are reflected by interspike-intervals in particular integrate-and-fire models.

However, in view of applications we shall also weaken the assumption that the input function is periodic and continuous. It turns out that many of the required properties of the firing map for most commonly used models, Perfect Integrator $\dot{x} = f(t)$ and Leaky Integrate-and-Fire $\dot{x} = -\sigma x + f(t)$, still hold if f is only locally integrable and almost periodic, either uniformly or in a sense of Stepanov ([5]).

References

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