Quasisymmetric rigidity of real maps

Happy birthday Lluis

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joint with Trevor Clark

Imperial College London

October 3, 2014

The maps that we are iterating.

$f : [0,1] \to [0,1]$ or $S^1 \to S^1$ that are C^3 and satisfy some extra conditions.

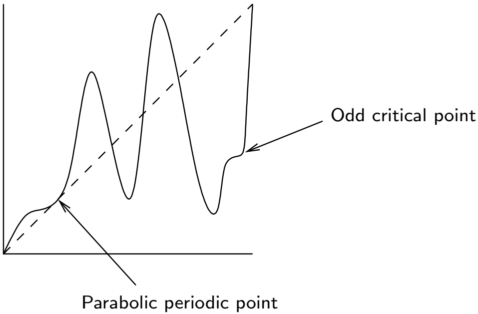

Figure : the type of map we will consider

Types of questions in one-dimensional dynamics

Lluis reminded me yesterday evening that we first met in Kiev in 1991, and that we had so much fun talking together.
Like Lluis, my work has mainly focused on one-dimensional dynamics.

Unfortunately, Lluis and I never wrote a paper together: our work focused on different aspects in 1-dimensional dynamics:

- combinatorial (Lluis)
- metric (my work)

This talk aims to convince Lluis that there also many fun results in the metric theory of 1-dimensional dynamics.

Specifically, I want to talk about global quasi-symmetric rigidity, namely the *Sullivan*'s programme.

Aim: Complete Sullivan's quasi-symmetric rigidity programme

A homeomorphism $h\colon [0,1] \to [0,1]$ is called **quasi-symmetric** (often abbreviated as *qs*) if there exists $K < \infty$ so that

$$\frac{1}{K} \leq \frac{h(x+t) - h(x)}{h(x) - h(x-t)} \leq K$$

for all $x - t, x, x + t \in [0,1]$. $(\implies$ Hölder; has h qc extension to $\mathbb{C})$.

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Sullivan's programme: prove that f is **quasi-symmetrically rigid**, i.e.

$$f, \tilde{f} \text{ is topologically conjugate } \implies$$

$$\tilde{f}, f \text{ are quasi-symmetrically conjugate.}$$

That is, any homeomorphism h with $h \circ f = g \circ h$ is **'necessarily' qs**.

Remark about Sullivan's aim:

- Quasi-symmetric maps have a quasiconformal extension to $\mathbb{C}$.
- **Sullivan's aim:** $\mathcal{C}$ should be **infinite dimensional Teichmuller space** with metric $d(f,\tilde{f}) = \inf K_h$ where K_h is the dilatation of qc extension $H\colon \mathbb{C} \to \mathbb{C}$ of qs conjugacy $h\colon N \to N$ between f and $\tilde{f}$.

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- Define $f \sim \tilde{f}$ when f are smooth conjugate. Is d a metric on $\mathcal{C}/\sim$?
- Yes (in the unimodal case, and probably also in the multimodal case). Indeed:
  $d(f,\tilde{f}) = 0 \implies$
  multipliers at corresponding periodic point of $f,\tilde{f}$ are equal $\implies$
  if $f,\tilde{f}$ unimodal, they are C^3 conjugate (by result of Li-Shen).

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- Current project: endow $\mathcal{C}/\sim$ with manifold structure.

Completing Sullivan's qs-rigidity programme

Theorem (Clark-vS)

Let $N = [0,1]$ *or* $N = S^1$*. Suppose* $f, \tilde{f} : N \to N$ *are* **topologically conjugate** *and are in* $\mathcal{C}$ *with* **at least one critical point***. Moreover, assume that the topological conjugacy is a bijection between*

- *the sets of critical points and the orders of corresponding critical points are the same, and*
- *the set of parabolic periodic points.*

Then f *and* $\tilde{f}$ *are* **quasisymmetrically conjugate***.*

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- This completes a programme initiated in the 80's by
  - **Sullivan** for interval maps: in his work on renormalisation;
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- The result is optimal, in the sense that no condition can be dropped.
- When $N = S^1$, the assumption $\exists$ critical point implies $\exists$ periodic point.

Real versus complex methods

The space $\mathcal{C}$ consists of real interval maps, and includes

- all real analytic maps;
- all C^∞ maps with finitely many critical points of integer order;
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Assume C^3 because then f extends to a C^3 map $F\colon U \to \mathbb{C}$ with U neighbourhood of I in $\mathbb{C}$, so that F is **asymptotically holomorphic** of order 3 on I; that is,

$$\frac{\partial}{\partial \bar{z}} f(x,0) = 0, \text{ and } \frac{\frac{\partial}{\partial \bar{z}} f(x,y)}{|y|^2} \to 0$$

uniformly as $(x,y) \to I$ for $(x,y) \in U \setminus I$.

Class of maps, $\mathcal{C}$

- $\exists$ finitely many critical points $c_1, \ldots, c_b$,
- $x \mapsto f(x)$ is C^3 when $x \neq c_1, \ldots, c_b$
- near each critical point $c_i, 1 \leq i \leq b$, we can express

$$f(x) = \pm|\phi(x)|^{d_i} + f(c_i),$$

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- **extra regularity** near **parabolic** periodic points.
  - Let $\lambda \in \{-1, 1\}$ be multiplier and s the period of p, then $\exists n$ with

$$f^s(x) = p + \lambda(x-p) + a(x-p)^{n+1} + R(|x-p|), R(|x-p|) = o(|x-p|^{n+1})$$

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The extra regularity makes it possible to use the Taylor series of f to study the local dynamics near the parabolic periodic points.

Why is qs-rigidity useful?

QS (QC) rigidity plays a crucial role in the following results:

- Density of hyperbolic maps (maps where each critical point converges to an attracting periodic point) (Lyubich, Graczyck-Świątek, Kozlovski, Shen, Kozlovski-Shen-vS).

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- Hyperbolicity of renormalization (Lyubich, Avila-Lyubich). (**Multimodal Palis conjecture: work in progress**).
- Monotonicity of entropy for real polynomial multimodal maps (Bruin-vS) and trigonometric families (Rempe-vS).

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- So methods require a mixture of real and complex tools.
- One of the main ingredients, **complex bounds**, fails for general complex maps.

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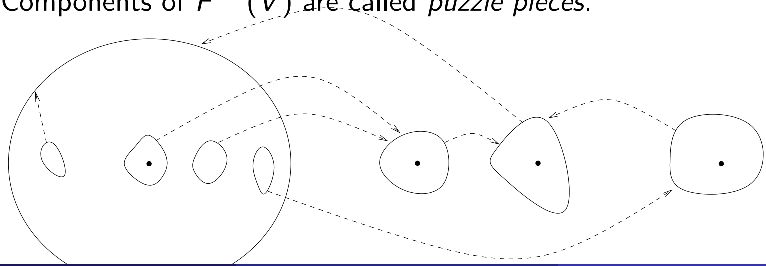
- Each component of U is mapped as a branched covering onto a component of V, and components of U are either compactly contained or equal to a component of V.
- Components of $F^{-n}(V)$ are called *puzzle pieces*.



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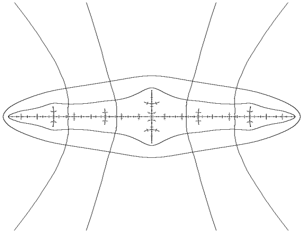
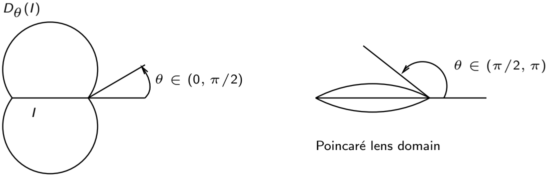
- In the non-renormalizable case one can repeatedly take first return maps to central domains.
- In the infinitely renormalizable case one has to start from scratch again and again (note: there is no straightening theorem when f is not holomorphic).



Poincaré disks and their diffeomorphic pullbacks

Let d be Poincaré metric on $\mathbb{C}_I := (\mathbb{C} - \mathbb{R}) \cup I$. A Poincaré disk is a set of the form $\{z; d(z, I) \leq d_0\}$ and is bounded by the union of two circle segments. These are used to construct a "Yoccoz puzzle" by hand.



- If f is polynomial with only real critical points and $f : J \to I$ a diffeomorphism: no loss of angle when pulling back $D_\theta(I)$ (by the Schwarz inclusion lemma).
- If f is real analytic or only C^3 one looses angle, whose amount depends on the size of $|I|^2$. One therefore needs to control this term along a pullback.

Poincaré disks and their pullbacks through critical points

- If $f: J \to I$ has a unique critical point then one looses more angle:

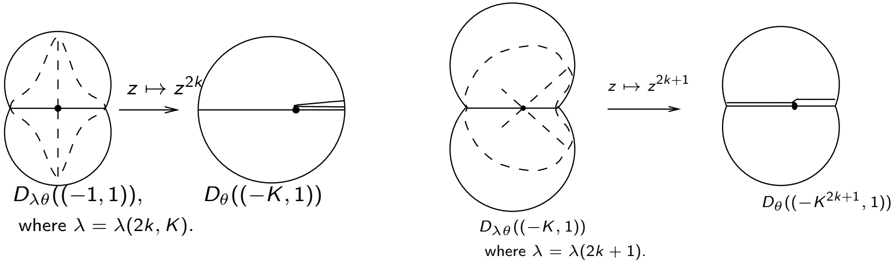


Figure : Inverses of Poincaré disks through a critical point

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- Clark-Trejo-vS: something similar even for C^3 maps, but then F is only asymptotically holomorphic.

Proving complex bounds

- From the **enhanced nest** construction (see next •) and a **remarkable result due to Kahn-Lyubich**, given a non-renormalizable complex box mapping at one level, one can obtain **complex box mappings**

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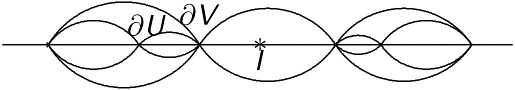
- Other choices will not give complex bounds, in general.
- In the **renormalizable case** and also in the C^3 case, the construction of complex box mappings and the proof of complex bounds is significantly *more involved*.
- Critical points of odd order require quite a bit of additional work.

Touching box mappings

If $f, \tilde{f} \in \mathcal{C}$ are topologically conjugate, the role of the Böttcher coordinate for polynomials is played by the construction of an "external conjugacy" between touching box mappings F_T and $\tilde{F}_T$.



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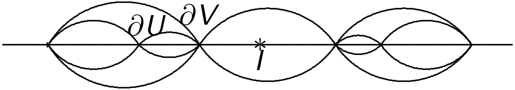

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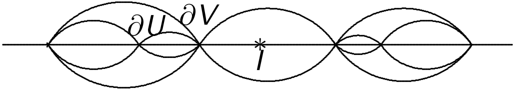

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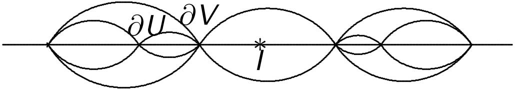

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- Real trace of the range contains a neighbourhood of the set of critical points, immediate basins of attracting cycles, and covers the interval.
- Used to pullback qc-conjugacies through branches that avoid $\mathrm{Crit}(f)$.

Idea for proving quasi-symmetric rigidity

- Using the **complex bounds** and a *methodology for constructing quasi-conformal homeomorphisms* (building on papers of Kozlovski-Shen-vS and Levin-vS), we construct quasi-conformal pseudo conjugacies on **small scale**.

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Remarks:

- In the C^3 case f, g have *asymptotically holomorphic extensions* near $[0, 1]$. Issue to deal with: arbitrary high iterates of f and g are not necessarily close to holomorphic.

A remark about qs-rigidity of critical circle homeomorphisms

The following result follows from work of de Faria-de Melo.

Theorem (follows from: de Faria-de Melo who use a result of Yoccoz)

Suppose that $f, \tilde{f} : S^1 \to S^1$ are critical circle homeomorphisms with irrational rotation number and one critical point. If $h : S^1 \to S^1$ is a homeomorphism such that $h \circ f = \tilde{f} \circ h$, then h is quasisymmetric.

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We have more:

Theorem (Clark-vS)

Suppose that $f, \tilde{f} \in \mathcal{C}$ are topologically conjugate critical circle homeomorphisms, then f and $\tilde{f}$ are quasisymmetrically conjugate.

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A third application will *hopefully* be a resolution of the *1-dimensional Palis conjecture* in full generality.

Application 1: Hyperbolic maps

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- **Every hyperbolic map** satisfying an additional transversality condition, that no critical point is eventually mapped onto another critical point, is **structurally stable**. (A nearby map is *topologically conjugate*, i.e. same up to topological coordinate change.)

Hyperbolic one-dimensional maps are dense

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This solves one of Smale's problems for the 21st century.

Density of hyperbolicity for real transcendental maps

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Remarks:

- Hence, density of hyperbolicity within the famous **Arnol'd family** and within space of trigonometric polynomials.
- Result implies conjectures posed by de Melo-Salomão-Vargas.

Hyperbolicity is dense within generic families

Theorem (vS: Hyperbolicity is dense within generic families)

For any (Baire) C^∞ generic family $\{g_t\}_{t\in[0,1]}$ of smooth maps:

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Question: What if f_0 and f_1 are '**totally different**'?

Density of hyperbolicity on $\mathbb{C}$?

Density of hyperbolicity for rational maps (Fatou's conjecture) is wide open. By Mañé-Sad-Sullivan it follows from:

Conjecture

If a rational map carries a measurable invariant line field on its Julia set, then it is a Lattès map.

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In the first case, the Julia set of course does not carry measurable invariant line field.

Strategy of the proof: local versus global perturbations

One approach: take g to be a **local perturbation** of f, i.e. find a 'bump' function h which is small in the C^k sense so that $g = f + h$ becomes hyperbolic.

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- In the C^2 **category** this approach has proved to be **unsuccessful** (but Blokh-Misiurewicz have partial results). Shen (2004) showed C^2 density using qs-rigidity results.

Proving density of hyperbolicity for $z^2 + c$

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Using a slightly more sophisticated argument, Kozlovski-Shen-vS also obtain that quasi-symmetric rigidity implies density of hyperbolicity when there are more critical points.

Application 2: monotonicity of entropy

In the early 90's, Milnor posed the

Monotonicity Conjecture. The set of parameters within a family of real polynomial interval maps, for which the topological entropy is constant, is connected.

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- Milnor-Tresser (2000) proved conjecture for cubics using
  - planar topology (in the cubic case the parameter space is two-dimensional) and
  - density of hyperbolicity for real quadratic maps.
- Bruin-vS: the set of parameters corresponding to polynomials of degree $d \geq 5$ with constant entropy is in general NOT locally connected.

Monotonicity of entropy: the multimodal case

Given $d \geq 1$ and $\epsilon \in \{-1, 1\}$, let P_ϵ^d space of

1. real polynomials $f\colon [0,1] \to [0,1]$ of degree $= d$;
2. all critical points in $(0, 1)$;
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Theorem (Monotonicity of Entropy)

For each integer $d \geq 1$, each $\epsilon \in \{-1,1\}$ and each $c \geq 0$,

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- Rempe-vS $\implies$ top. entropy of $x \mapsto a\sin(x)$ monontone in a.