Uniquely minimal spaces as examples of rigid-like spaces in topological dynamics

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1. Rigid spaces for homeomorphisms
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6. Idea of a construction of uniquely minimal spaces

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- G countable $\Rightarrow$ X exists in the class of Peano continua of any $\dim > 0$ (**de Groot, Wille 1958**)

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In particular (in the class of Peano continua):

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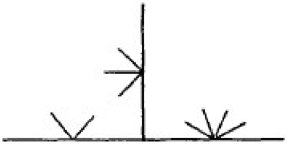
First **examples** of rigid spaces (1-dim Peano continua) for homeomorphisms (**de Groot, Wille 1958**):

- dendrite with a dense set of branching points of different orders



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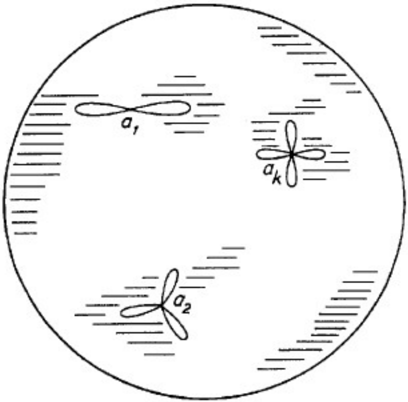

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What about $C(X,X) \setminus$ {constant maps}? In general it is not a monoid (composition of non-constant maps may be a constant map). Nevertheless, in some cases it is a monoid.

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In particular:

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3. Cook continua as extremely rigid spaces

$\exists$ nondegenerate metric continuum C (**Cook continuum**) such that for every subcontinuum K and every continuous map $f : K \to C$, either f is constant (i.e. $f(K)$ is a singleton) or $f(x) = x$ for all $x \in K$ (hence $f(K) = K$).

- $\exists C =$ 1-dim, hence embeddable into $\mathbb{R}^3$ (**Cook 1967**)
- $\exists C =$ chainable (i.e. arc-like), hence planar non-separating continuum (**Maćkowiak 1986**)

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Optimist: "Such spaces show us important restrictions. For instance, among metric continua there are spaces rigid for continuous maps and therefore one cannot hope that every continuum admits a continuous map with interesting dynamics, say with an omega-limit set different from a singleton."
Pessimist: "Nobody takes care about such degenerate dynamics."

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Optimist: "Well,

- rigid spaces can be used to build spaces which are not really rigid, but still can be called **rigid-like** (i.e. they admit a relatively small number of continuos maps/homeos) and they are useful to disprove quite serious conjectures;
- spaces which are **rigid-like w.r.t. some dynamical property** (i.e. spaces admitting a small number of continuos maps/homeos with that property, or even a small number of continuos maps/homeos at all and they all have that property) sometimes appear as by-products, or we construct them on purpose, to be able to answer natural 'dynamical' questions."

Pessimist: "Examples?"

4. Are rigid spaces useful/meaningful in topological dynamics?

Optimist:

- "Cook continua can be used to produce a rigid-like space which admits only a small number of continuous maps (and we are able to describe them) and serves as a counterexample to a conjecture in the theory of topological sequence entropy (in preparation, **X. Ye, R. Zhang and Ľ. S.**).
- **Bruin, Štimac (2012)** showed that, for some tent maps, if we restrict all self-homeomorphisms of the inverse limit space to its core (which is a chainable indecomposable continuum), then we get just all the iterates of the shift homeomorphism. So, they all, except of the identity, are transitive. This space can be called "transitivity" rigid-like for homeomorphisms.
- Moreover, we are going to explain that there exist **uniquely minimal spaces**:"

5. Existence of uniquely minimal spaces and applications

For a compact metric space X there are two possibilities:

- X does not admit any minimal homeomorphism
- X admits a minimal homeomorphism
  (in this case, if X is infinite then in known examples usually (always?) X admits uncountably many homeomorphisms and even uncountably many of them are minimal)

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Answer. Yes, in the following sense.

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Definition. An infinite compact metric space X is **Slovak** if it is **uniquely minimal** in the following sense: X admits a minimal homeomorphism T and $H(X, X) = \{T^n : n \in \mathbb{Z}\}$.

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Observation.

- The assumption that X is infinite eliminates two trivial examples: the one-point space and the two-point space.
- If X is Slovak then $\mathrm{card}\, X = c$, X has no isolated point and all iterates T^n, $n \in \mathbb{Z}$ are different, i.e. $\mathbf{H(X,X)} \approx \mathbb{Z}$. Moreover, **all iterates T^n, except identity, are minimal**.

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Theorem. There exist Slovak spaces in the class of metric continua. (Moreover, the topological entropies of generating homeomorphisms T exhaust the interval $[0, \infty]$.)

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Corollary 1. There exist infinite metric continua, which are not homeomorphic to the circle, admitting minimal homeomorphisms but not minimal non-invertible maps.

Idea of proof. Let X be a Slovak space constructed in the proof of Theorem. In that proof we show that every homeomorphism $X \to X$ coincides with the iterate of some distinguished minimal homeomorphism $X \to X$. Developing further these ideas, one can show that any minimal continuous map $X \to X$ is invertible.

5. Existence of uniquely minimal spaces and applications

(X, f) ... X compact metric, $f : X \to X$ continuous map

$(C(X,X), F_f)$ ... functional envelope of 1st kind,
$C(X,X)$ with uniform metric
$F_f(\varphi) = f \circ \varphi$ for all $\varphi \in C(X,X)$
easy: $h_{\mathrm{top}}(F_f) \geq h_{\mathrm{top}}(f)$

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(X, f) ... X compact metric, $T : X \to X$ homeomorphism

$(H(X, X), F_T)$ ... functional envelope of 2nd kind
$H(X, X)$ with uniform metric
$F_T(\phi) = f \circ \phi$ for all $\phi \in H(X, X)$
question (**Kolyada, Semikina 2013**): Is

$$h_{\text{top}}(F_T) \geq h_{\text{top}}(T) \text{ ?}$$

5. Existence of uniquely minimal spaces and applications

Corollary 2. There exists a dynamical system given by a compact metric space X and a homeomorphism $T : X \to X$ with finite positive or even infinite entropy, whose functional envelope $(H(X, X), F_T)$ has entropy zero.

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Idea of proof. Let (X, T) be a Slovak space with T being the generating homeomorphism of the group $H(X,X)$. We know that T may have positive, even infinite entropy. However, $H(X,X)$ is only countable and this enables to show that $h_{\mathrm{top}}(F_T) = 0$.

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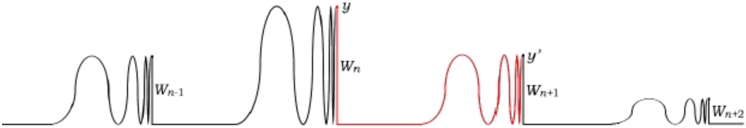
Step 4 (technical step).

- the continuum X has uncountably many composants (orbits of the flow Φ); choose a composant γ and a point $x_0 \in \gamma$.
- on a closed arc around x_0 (minus the point x_0 itself) and lying in γ, we define a function which looks like a one-sided topologist's sine curve (values in $[0,1]$, wiggles of height 1 in any left neighbourhood of x_0, constant value 0 to the right of x_0). It is continuous and not defined at x_0.
- extend it to a continuous function $f : X \setminus \{x_0\} \to [0,1]$

6. Idea of a construction of uniquely minimal spaces

- let $F = \sum_{n\in\mathbb{Z}} a_n f \circ T^n$, where the coefficients a_n are all strictly positive, $\sum_{n\in\mathbb{Z}} a_n = 1$ and satisfying some technical assumptions (F is defined on X minus the T-orbit of x_0).
- then one can show that both the mapping $(x, F(x)) \mapsto (Tx, F(Tx))$ and its inverse are uniformly continuous homeomorphisms of the graph of F. Therefore, the map $(x, F(x)) \mapsto (Tx, F(Tx))$ extends to a homeomorphism $\bar{T}$ (=notation) of $\bar{F}$ (=the closure of the graph of F).
- $\bar{F} \subseteq X \times [0, 1]$ is our **Slovak space**, looks as follows:

  the composant $\bar{\gamma}$ of $\bar{F}$ "above" γ has basicly this shape:



the other composants of $\bar{F}$ are continuous bijective images of the real line

6. Idea of a construction of uniquely minimal spaces

If φ is an arbitrary homeomorphism $\bar{F} \to \bar{F}$ then:

- φ sends path components to path components preserving the type (real line or closed half-line)
- the set Z of upper endpoints of the vertical intervals W_n is therefore preserved by φ
- φ preserves the successor relation on Z
- therefore, on Z, φ coincides with $\bar{T}^n$ for some n. Since Z is dense in $\bar{F}$ (being the orbit of the minimal homeomorphism $\bar{T}$), we get that φ coincides with $\bar{T}^n$ everywhere on $\bar{F}$. Q.E.D.