Universality and zero-entropy systems

New Perspectives in Discrete Dynamical Systems

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What is the meaning of **universality**?

All measure-theoretic systems are assumed standard.

Definition

A topological dynamical system (Y, S) is **universal** in a class $\mathcal{T}$ of measure-theoretic systems if the following two conditions hold:

1. For any S-invariant measure ν on Y the measure-theoretic system (Y, ν, S) belongs to $\mathcal{T}$, and
2. For any system $(X, \mu, T) \in \mathcal{T}$ there exists an invariant measure ν on Y such that (Y, ν, S) is measure-theoretically isomorphic to (X, μ, T).

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Informally: The simplex of invariant measures of (Y, S) contains nothing but the isomorphic copies of measures of systems from $\mathcal{T}$.

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PROBLEM (Weiss): Find a universal model for the class of automorphisms of zero entropy, or more generally, for the class $\mathcal{T} = \{T : h(T) \leq r\}$, for $r \geq 0$.

Theorem

There does not exist a topological dynamical system, universal in the class of all automorphisms of entropy zero.

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- (measure-theoretic complexity, Ferenczi 1997) a simple fact that in symbolic realizations of measure-theoretic ergodic systems this complexity is dominated by the *symbolic complexity*
- if $U(n) \nearrow \infty$ is a sequence for which $\frac{1}{n} \log U(n) \to 0$ then there exists an ergodic zero-entropy system whose measure-theoretic complexity grows faster than $U(n)$

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  Q a finite measurable partition of X, indexed by Λ,
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  $x_n = i \iff T^n(x) \in Q_i$
- A block $B \in \Lambda^n$ corresponds to a *cylinder set*
  $[B] = \{x : x[0, n-1] = B\}$.
  The *Hamming distance* between two cylinders $B, C \subset \Lambda^n$ is

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$$d_H(B, C) = \frac{1}{n}\text{card}\{i = 0, \ldots, n-1 : b_i \neq c_i\}.$$

- For $\epsilon > 0$, the (ϵ, n)- ball with center $B \in \Lambda^n$ is:

$$\mathbb{B}_H(B, \epsilon) = \bigcup_{\{C \in \Lambda^n : d_H(B,C) < \epsilon\}} [C].$$

- Define $K_\mu(Q, \epsilon, n, T)$ to be the minimal number of (ϵ, n)- balls, needed to cover a part of the space X of measure μ at least $1 - \epsilon$.

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- If $U(n)$ is a nondecreasing sequence of positive numbers satisfying

$$\lim_{\epsilon\to 0} \limsup_{n\to\infty} \frac{K_\mu(Q, \epsilon, n, T)}{U(n)} \leq 1$$

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  then we say that *the measure-theoretic complexity* of the system (X,μ,T) with regard to the partition Q *is dominated* by $U(n)$.
- A simple fact: if (Y,S) is a symbolic system with (symbolic) complexity $U_Y(n)$, ν is an S-invariant ergodic measure, P is a finite partition then clearly $K_\nu(P,\epsilon,n,S)\leq U_Y(n)$.

Third ingredient again:

Lemma (Rapidly growing complexity in entropy zero)

For every nondecreasing sequence $U(n)$ of positive numbers such that $\frac{1}{n}\log U(n) \to 0$, there exists an ergodic measure-theoretic dynamical system (X, μ, T) with $h_\mu(T) = 0$, and a finite measurable partition Q of X such that the measure-theoretic complexity of (X, μ, T) with regard to Q is NOT dominated by $U(n)$.

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- First ingredient: (Y', S') has a symbolic extension (Y, S) of zero entropy. Let $(U_Y(n))_{n\geq 1}$ be the *symbolic complexity* of (Y, S).
- By the second ingredient, if ν is an S-invariant ergodic measure and P is a finite partition of Y, then the measure-theoretic complexity of (Y, ν, S) with respect to P is dominated by $U_Y(n)$.

- As $h_{top}(S) = 0$, we have $\frac{1}{n} \log U_Y(n) \to 0$, so by the third ingredient, there exists an ergodic zero-entropy measure-preserving system (X, μ, T), and a finite partition Q of X, such that the measure-theoretic complexity of (X, μ, T) with regard to Q grows essentially faster than $U_Y(n)$.

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- In particular, the measure-theoretic complexity of (Y, ν, S) with regard to P is NOT dominated by $U_Y(n)$. This contradicts the existence of a universal zero-entropy system.

- Elements of X are arrays, rows $1, 2, \ldots$ contain an appropriate odometer.
  To define the 0th row need two elementary operations.
  If $\mathcal{C}$ is a collection of blocks of a fixed length and $q \in \mathbb{N}$ then $\mathcal{C}^q$ is the family of all (independent) concatenations of q blocks from $\mathcal{C}$.
  Also, $\mathcal{C}_{\text{rep}} = \{CC : C \in \mathcal{C}\}$ is the collection of repetitions of blocks from $\mathcal{C}$.
  Clearly $\mathrm{card}(\mathcal{C}_{\text{rep}}) = \mathrm{card}(\mathcal{C})$ and $\mathrm{card}(\mathcal{C}^q) = (\mathrm{card}\, \mathcal{C})^q$.

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- We determine X by requiring that $x \in X$, if and only if, for any k, every k-block of x belongs to $(\mathcal{B}_k)_{\text{rep}}$. In other words, $x \in X$ if 0th row x_0 is, for every k, an infinite concatenation of the blocks from $(\mathcal{B}_k)_{\text{rep}}$ with "gluing points" at the k-markers.

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- Define an invariant measure on X: for each $k \geq 1$, declare all k-blocks occurring in X to have the same measure:

$$\mu(\{x : x_0[0, N_k - 1] = BB,\ x_k(0) = 1\} = \frac{1}{N_k \cdot \operatorname{card}(\mathcal{B}_k)} = \frac{1}{N_k 2^{p_k}},$$

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  for all $B \in \mathcal{B}_k$.
- Verification that the above indeed determines a shift-invariant ergodic measure on X is standard. Variational principle implies that $h_\mu(T) = 0$.

- Let $Q = \{[0], [1]\}$, where $[0] = \{x \in X : x_0(0) = 0\}$ and $[1] = \{x \in X : x_0(0) = 1\}$, be the zero-coordinate partition of the 0th row, lifted to X.

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- X splits into N_k sets $P_0, P_1, P_2, \ldots, P_{N_k-1}$ of equal measure $\frac{1}{N_k}$, where $P_i = \{x : x_k(i) = 1\}$, for $i = 0, \ldots, N_k - 1$, and $k \geq 1$.

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- The mechanism of concatenations and repetitions:
  the entries of a block $B \in \mathcal{B}_k$ are determined by a subset of p_k (out of $N_k/2$) coordinates, making up a fraction $\frac{1}{2^k}$ of the length of B. The symbols occurring along these coordinates are arbitrarily, each of them is then repeated 2^k times in B.

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- We have a bijection $\phi_k : \mathcal{B}_k \to \{0,1\}^{p_k}$ such that:
  - ϕ_k is "distance-preserving", i.e. $d_H(B, C) = d_H(\phi_k(B), \phi_k(C))$,
  - $\lambda(\phi_k(B)) = \mu(B|P_0) = 2^{-p_k}$, where $\lambda = \{\frac{1}{2}, \frac{1}{2}\}^{\mathbb{Z}}$.

  This bijection allows us to perform our estimation using the measure λ.

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- Notice that applying Stirling's formula we have

$$\mathrm{card}\{C \in \{0,1\}^n : d_H(B,C) < \delta\} = \sum_{j<\delta n} \binom{n}{j} \leq 2^{nH(2\delta)},$$

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- Fix $\epsilon > 0$. All 2^n cylinders of length n have equal measure λ, so the minimal cardinality of a subfamily of cylinders whose elements cover up more than $1-\epsilon$ (in measure) of the space, must be at least $(1-\epsilon)\cdot 2^n$. All (δ, n)-Hamming balls include the same number of cylinders, so we have

$$M(\delta, n, \epsilon) \geq \frac{(1-\epsilon)\cdot 2^n}{2^{nH(2\delta)}} = (1-\epsilon)2^{n(1-H(2\delta))} \tag{1}$$

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  (for all n sufficiently large).
- Using the above we can estimate from below the quantity $K_\mu(Q, \epsilon, N_k, T)$.