No Semiconjugacy to a Map of Constant Slope

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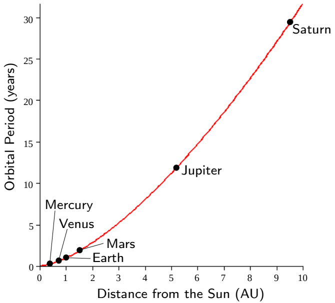
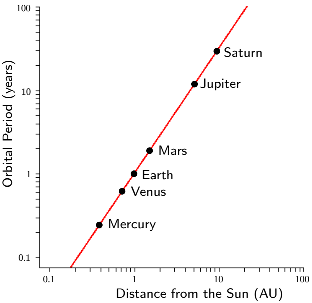
Joint work with Michał Misiurewicz

Kepler's Third Law





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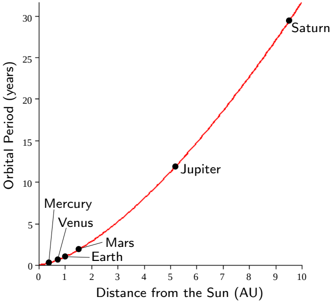
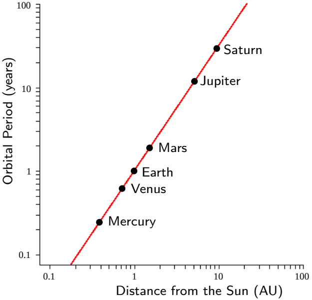
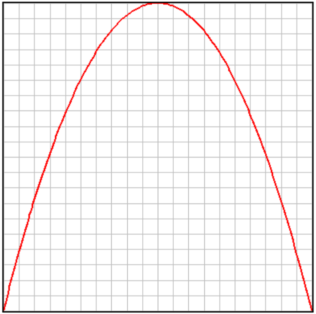


$$T = D^{3/2}$$

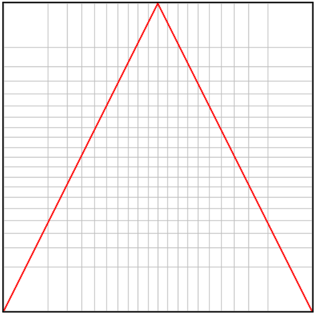


$$\log T = \frac{3}{2} \log D$$

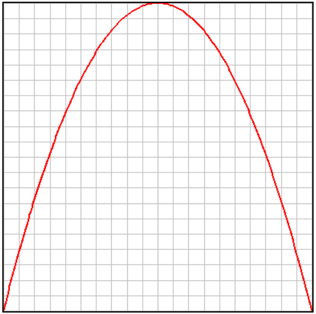
Constant Slope



f - Logistic Map



g - Tent Map



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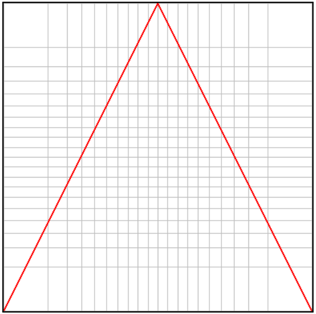


g - Tent Map

$$g = \psi \circ f \circ \psi^{-1}, \quad \psi(x) = \frac{\arccos(1-2x)}{\pi}$$

Conjugacy to a Map of Constant slope

Theorem (Parry, 1966)

Let $f : [0,1] \to [0,1]$ be continuous, piecewise monotone, and topologically transitive. Then f is conjugate to an interval map of constant slope.

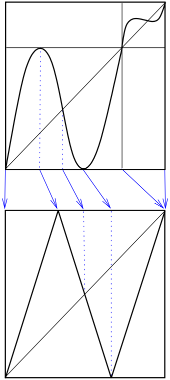
$$\begin{tikzcd} {[0,1]} \arrow[r, "f"] \arrow[d, "\psi"'] & {[0,1]} \arrow[d, "\psi"] \\ {[0,1]} \arrow[r, "g \text{ (constant slope)}"'] & {[0,1]} \end{tikzcd}$$

Semiconjugacy to a Map of Constant Slope

Theorem (Milnor-Thurston, 1988)

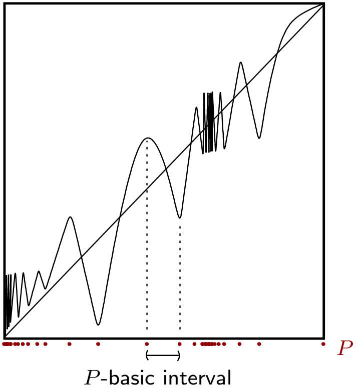
Let $f : [0,1] \to [0,1]$ be continuous, piecewise monotone, and have positive topological entropy. Then f is semiconjugate to an interval map of constant slope. This semiconjugating map is nondecreasing.

$$\begin{tikzcd} {[0,1]} \arrow[r, "f"] \arrow[d, "\psi"'] & {[0,1]} \arrow[d, "\psi \text{ (nondecreasing)}"] \\ {[0,1]} \arrow[r, "g \text{ (constant slope)}"'] & {[0,1]} \end{tikzcd}$$



Countably Piecewise Monotone

Research Question: What about maps with infinitely many turning points?



A continuous map $f : [0,1] \to [0,1]$ is called *countably piecewise monotone* if the closure of the set of turning points is countable.
Notation: P - a closed countable set containing the turning points.

Countably Piecewise Monotone and Markov

Markov Condition: P can be chosen such that $f(P) \subseteq P$

Theorem (Bobok, 2012)

Suppose f is countably piecewise monotone and Markov w.r.t. P. Then

There exists a nondecreasing semiconjugacy of f with a map of constant slope λ $\iff$ *The 0-1 transition matrix for (f, P) admits a summable, nonnegative eigenvector of eigenvalue λ*

Countably Piecewise Monotone and Markov

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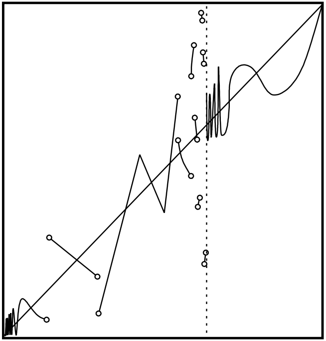
Research Goals:

- Produce non-trivial examples that violate the criterion.
- Tackle the non-Markov case.

Countably Piecewise Monotone Piecewise Continuous



We study the class $\mathcal{C}$ of maps f such that

- there exists a closed, countable set P such that f is continuous and strictly monotone on each P-basic interval.
- $f : [0,1] \setminus P \to [0,1]$
- two maps in $\mathcal{C}$ are considered equal if they differ only on a countable, closed set.

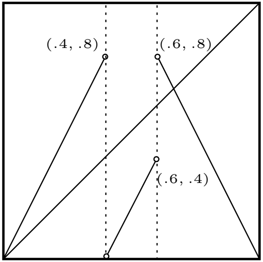
The Pullback Operator f^*

$\mathcal{M}$ - the set of nonatomic, Borel measures on $[0, 1]$.

Definition

To each $f \in \mathcal{C}$ associate the operator $f^* : \mathcal{M} \to \mathcal{M}$ given by

$$(f^*\mu)(A) = \sum_I \mu(f(I \cap A))$$



Example:

f - piecewise linear, constant slope 2

m - Lebesgue measure

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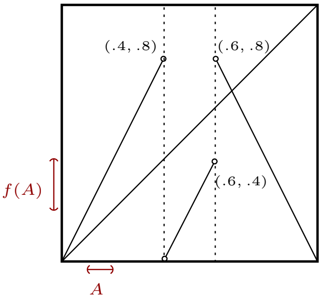

Example:

f - piecewise linear, constant slope 2

m - Lebesgue measure

$A = (.1, .2)$

$m(A) = .1$

$(f^*m)(A) = m(f(A)) = .2$

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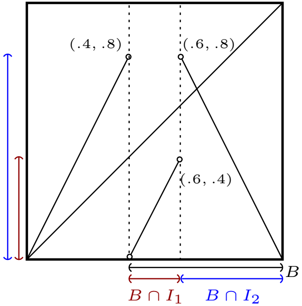

Example:

f - piecewise linear, constant slope 2

m - Lebesgue measure

$B = (.4, 1)$

$m(B) = .6$

$(f^*m)(B) = .4 + .8 = 1.2$

Criterion for Semiconjugacy

Lemma

The map $f \in \mathcal{C}$ has constant slope λ iff the Lebesgue measure m satisfies $f^* m = \lambda m$.

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Theorem (Criterion for Semiconjugacy)

Let $f \in \mathcal{C}$ and fix $\lambda > 0$. Then

There exists a nondecreasing semiconjugacy of f with a map of constant slope λ	$\iff$	*There exists a probability measure $\mu \in \mathcal{M}$ such that $f^* \mu = \lambda \mu$*

Criterion for Semiconjugacy

Lemma

The map $f \in \mathcal{C}$ has constant slope λ iff the Lebesgue measure m satisfies $f^*m = \lambda m$.

Theorem (Criterion for Semiconjugacy)

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There exists a nondecreasing semiconjugacy of f with a map of constant slope λ	$\iff$	*There exists a probability measure $\mu \in \mathcal{M}$ such that $f^*\mu = \lambda\mu$*

Sketch of Proof.

($\Rightarrow$) Define $\mu = \psi^* m$.
($\Leftarrow$) Define ψ by $\psi(x) := \mu([0, x])$. Then $\psi_* \mu = m$. ■

No Semiconjugacy

Theorem (Main Technical Theorem)

Let $f \in \mathcal{C}$ *and fix* $\lambda > 2$.
Suppose there is an infinite measure $\mu \in \mathcal{M}$ *such that* $f^*\mu = \lambda\mu$.
$+$ *technical hypothesis*
$+$ *technical hypothesis*
Then there is no probability measure $\nu \in \mathcal{M}$ *such that* $f^*\nu = \lambda\nu$.

No Semiconjugacy

Theorem (Main Technical Theorem)

Let $f \in \mathcal{C}$ and fix $\lambda > 2$.
Suppose there is an infinite measure $\mu \in \mathcal{M}$ such that $f^*\mu = \lambda\mu$.
 Assume f is substantially transitive.
 Assume $\exists P, \exists \delta > 0$, $\forall P$-basic interval I, $\delta \leq \mu(I) < \infty$.
Then there is no probability measure $\nu \in \mathcal{M}$ such that $f^*\nu = \lambda\nu$.

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Proof - Part I.

We show the ergodicity of the measure μ.
Assume E is invariant, $\mu(E) > 0$.
"Lebesgue Density Theorem" $\Rightarrow$ E has a μ-density point x.
Technical Hypothesis 2 $\Rightarrow$ $\exists$ successively smaller intervals $L_k \ni x$ with "large" monotone continuous images $f^{n_k}(L_k)$, each of measure δ.
E has the same density in these images as in the sets L_k.
We find an open interval U (of measure δ) in which the density of E is 1.
"Substantial Transitivity" $\Rightarrow$ $E = [0, 1]$ (mod 0).

No Semiconjugacy

Theorem (Main Technical Theorem)

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Suppose there is an infinite measure $\mu \in \mathcal{M}$ such that $f^*\mu = \lambda\mu$.
Assume f is substantially transitive.
Assume $\exists P, \exists \delta > 0$, $\forall P$-basic interval I, $\delta \leq \mu(I) < \infty$.
Then there is no probability measure $\nu \in \mathcal{M}$ such that $f^*\nu = \lambda\nu$.

Proof - Part II.

Now assume that ν exists.
After replacing μ by $\mu + \nu$, we obtain absolute continuity $\nu << \mu$.
Let $\xi = d\nu/d\mu$ be the Radon-Nikodym derivative.
$\xi \circ f = df^*\nu/df^*\mu = (\lambda d\nu)/(\lambda d\mu) = \xi$.
Ergodicity $\Rightarrow$ ξ is constant.
Thus, a probability measure is a constant multiple of an infinite measure.
Absurd! ∎

No Semiconjugacy

Theorem (Main Theorem)

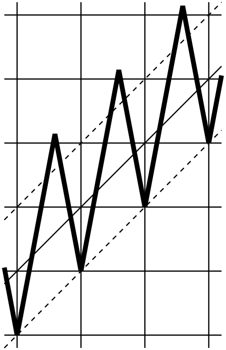
Assume that $F : \mathbb{R} \to \mathbb{R}$ is continuous, topologically transitive, has constant slope $\lambda > 1$, and is the lifting of a degree one circle map $f : \mathbb{R}/\mathbb{Z} \to \mathbb{R}/\mathbb{Z}$.

Assume also that f is piecewise monotone with finitely many pieces.

Take any homeomorphism $h : \mathbb{R} \to (0, 1)$ and let $g : [0, 1] \to [0, 1]$ be the continuous interval map $h \circ F \circ h^{-1}$ (with additional fixed points at 0, 1).

Then there does not exist any nondecreasing semiconjugacy of g to an interval map of constant slope.

Example



$F : \mathbb{R} \to \mathbb{R}$

No Semiconjugacy

Proof - Part I.

$$\begin{tikzcd}
(0,1) \arrow[r, "g"] \arrow[d, "h"'] & (0,1) \arrow[d, "h"] \\
\mathbb{R} \arrow[r, "F"'] \arrow[d, "\pi"'] & \mathbb{R} \arrow[d, "\pi"] \\
\mathbb{R}/\mathbb{Z} \arrow[r, "f"'] & \mathbb{R}/\mathbb{Z}
\end{tikzcd}$$

Suppose there is a semiconjugacy of g to an interval map of constant slope λ'.

$\exists \nu_g \in \mathcal{M}$, $g^* \nu_g = \lambda' \nu_g$.

Push down to the circle. $\nu_f := \pi_* h_* \nu_g$.

$f^* \nu_f = \lambda' \nu_f$.

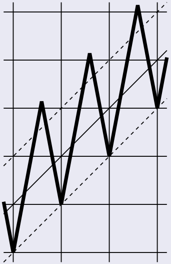
There is a nondecreasing semiconjugacy of f to a circle map of constant slope λ'.

By transitivity, that semiconjugacy is a conjugacy.

Entropy ($= \log$ of constant slope) is a conjugacy invariant. Therefore $\lambda' = \lambda$.

This rules out any slope except λ.

Proof - Part II.



$$F : \mathbb{R} \to \mathbb{R}$$

Constant Slope λ

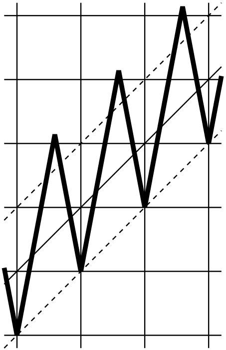
m - Lebesgue Measure

$$F^* m = \lambda m$$

$$\begin{tikzcd}
(0,1) \arrow[r, "g"] \arrow[d, "h"'] & (0,1) \arrow[d, "h"] \\
\mathbb{R} \arrow[r, "F"'] \arrow[d, "\pi"'] & \mathbb{R} \arrow[d, "\pi"] \\
\mathbb{R}/\mathbb{Z} \arrow[r, "f"'] & \mathbb{R}/\mathbb{Z}
\end{tikzcd}$$

Let $\mu = h^* m$. Thus, $\mu([0,1]) = \infty$.
The technical hypotheses are met and $g^* \mu = \lambda \mu$.
Therefore there is no probability measure $\nu \in \mathcal{M}$ such that $g^* \nu = \lambda \nu$.
Therefore there is no semiconjugacy to a map of constant slope λ. ∎

Remarks



$$F : \mathbb{R} \to \mathbb{R}$$

How are we using the extra structure from the underlying circle map?

- When F^* acts on prob. measures, the only possible eigenvalue is λ.
- Markov vs non-Markov ...
- Additional tools for studying entropy ...

Further Reading

Jozef Bobok,
Semiconjugacy to a Map of a Constant Slope,
Studia Math. **208** (2012), 213–228.

Jozef Bobok and Henk Bruin
Semiconjugacy to a Map of a Constant Slope II,
Forthcoming.

Michał Misiurewicz and Samuel Roth,
No Semiconjugacy to a Map of Constant Slope,
To Appear: Ergodic Theory and Dynamical Systems.
Arxiv 1403.2701.