Looking for two-dimensional strange attractors

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Tossa de Mar
Catalonia, october 2-4, 2014

Part I

Introduction

Discrete dynamics

For the present we will refer to iterations of C^k-maps or C^k-diffeomorphisms.

$$f : M \to M,$$

defined on a C^k-manifold M. Mostly, $M = \mathbb{R}^n$, $n = 1, 2, 3$
At some moment we will need to distinguish between invariant sets and strictly invariant sets.

Definitions

A subset $A \subset M$ of M is said invariant iff $f(A) \subseteq A$
A subset $A \subset M$ of M is said strictly invariant iff $f(A) = A$

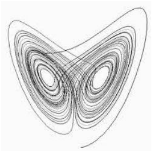
Definitions

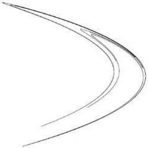
Attractor A compact invariant set Λ with a dense orbit (transitive) and whose stable set W^s has a non-empty interior.

Strange Λ contains a dense expansive orbit (with a positive Liapunov exponent): sensitive dependence on initial conditions.

Outstanding examples

- Lorenz Attractor (continuous systems)
- Hénon Attractor (discrete systems)





How can an invariant compact $\Lambda = f(\Lambda)$ be expansive?

- If f is a C^k-map then we can suppose that there exist non-empty subsets Λ_1 and Λ_2 of Λ such that
$$\Lambda = \Lambda_1 \cup \Lambda_2 \text{ and } f(\Lambda_1) = \Lambda.$$
- If f is a C^k-diffeomorphism then Λ may be a submanifold of M which folds more and more over itself. Λ is a set with fractional dimension.

Part II

One-dimensional strange attractors

Tent maps

The simplest example of strange attractor is provide by the following one-parametric family of tent maps:

$$f_\mu \quad : \quad x \in [0,1] \rightarrow [0,1]\,,\ \mu \in (1,2]$$

$$f_\mu(x) \quad = \quad \begin{cases} \mu x & , \ 0 \leq x \leq 1/2 \\ \mu - \mu x & , \ 1/2 \leq x \leq 1 \end{cases}$$

The invariant interval $[\mu(1-\mu/2), \mu/2]$ is an strange attractor of f_μ for $\mu > \sqrt{2}$.

Quadratic maps

Strange attractors can be also found for the following one-parameter family of quadratic maps

$$Q_a : x \in [-1,1] \to [-1,1], \ a \in (0,2]$$

$$Q_a(x) = 1 - ax^2.$$

In fact, Q_a is conjugate to the tent map f_μ for $a = \mu = 2$.
For $a < 2$, there exists a positive Lebesgue measure set $E \subset (2-\varepsilon, 2)$, with ε arbitrarily small, such that the interval $\left[Q^2(0), Q(0)\right]$ is an strange attractor: the critical orbit is dense in $\left[Q^2(0), Q(0)\right]$ and its Lyapunov exponent is positive.
Hence there exists an absolutely continuous invariant measure.

References

- **Benedicks, M.; Carleson, L.** *On iterations of* $1 - ax^2$ *on* $(-1,1)$. Ann. Math., 122, 1985.
- **Jacobson, M.** *Absolutely continuous invariant measures for one-parameter families of one-dimensional maps*. Comm. Math. Phys. 81, 1981.

Smooth unimodal maps

The existence of strange attractors can be also stated for families $f_a : I \to I$ of sufficiently smooth unimodal maps such that:

- f_a has a quadratic critical point c.
- f_a has a fixed point q_a in the boundary of I which is repelling.
- The map $(x, a) \to (f_a(x), Df_a(x), D^2 f_a(x))$ is C^1.
- There exists a value $a = a_*$ for wich f_{a_*} is a Misiurewicz map: the forward iterates of $f_{a_*}(c)$ remain outside a neighbourhood U of c.
- f_{a_*} has no periodic attractors.
- The transversality condition $\frac{d}{da}(x_a - f_a(c)) \neq 0$ holds for $a = a_*$, where $x_a \neq q_a$ is the point such that $f_a(x_a) = q_a$.

References

- **De Melo, W.; Van Strien, L.** *One-dimensional dynamics*. Springer Verlag, 1993.
- **Pumariño, A.; Rodríguez J.A.** *Coexistence and persistence of strange attractors*. Lecture Notes in Math. 1658. Springer Verlag, 1997.

Strange attractors for diffeomorphisms

Tent maps, quadratic maps or, in general, unimodal ones are not injective. Their iterations are not useful to model most of the processes: those called *reversible processes*.

The reversible processes are mathematically modeled by means of flows or by iterating diffeomorphisms. Therefore, concepts like strange attractor must be mainly set in the framework of diffeomorphisms.

An early example of diffeomorphism with an strange attractor was given by Smale: the "solenoid". It is a hyperbolic attractor and therefore is observable (structurally stable). However, there is no natural scenery in dynamical systems leading to the abundance of solenoids.

Since strange attractors are called to describe the nature of dissipative chaos, they should be observable and abundant in generic contexts.

Non-hyperbolic strange attractors

Strange attractors found numerically in applications, such as the Hénon attractor and Lorenz attractor, are not hyperbolic. But,

Does there exist any non-hyperbolic strange attractor which is somehow persistent?

Are these attractors abundant in any generic context?

The first proof of the existence of a non-hyperbolic strange attractor was given for the Hénon family by Benedicks and Carleson.
A first answer to the second question was given by Mora and Viana for a generic family of surface diffeomorphisms unfolding a homoclinic tangency.

References

- **Benedicks, M.; Carleson, L.** *The dynamics of the Hénon map*. Ann. Math., 133, 1991.
- **Mora, L.; Viana, M.** *Abundance of strange attractors*. Acta. Math. 171, 1993.

The Hénon family

The Hénon family is the two-parameter family of diffeomorphisms

$$H_{a,b}(x, y) = (1 - ax^2 + y, bx).$$

If $b > 0$ **is small enough then for a positive Lebesgue measure set** $E(a, b)$ **of values of** a **near to** $a = 2$**, the corresponding diffeomorphism** $H_{a,b}$ **exhibits a strange attractor.**
Positive Lebesgue measure of $E(a, b)$ means that the strange attractor is observable with positive probability. It is said that the attractor is **persistent in the sense of the measure**. Henceforth **persistent or probable**.

The limit family

When (a, b) is near to $(2, 0)$, the Hénon family has a saddle fixed point q close to the point $(-1, 0)$. **The Hénon attractor is the closure of the unstable manifold $W^u(q)$ of q.**
It is crucial to note that if b goes to $b = 0$ then this manifold $W^u(q)$ is pressed against the attractor of

$$H_{a,0}(x, y) = (1 - ax^2 + y, 0),$$

which is the attractor in $y = 0$ of the quadratic family

$$Q_a(x) = 1 - ax^2.$$

Definition

$\widetilde{Q}_a(x) = (1 - ax^2, 0)$ is the **limit family** of $H_{a,b}$ when $b \to 0$ or, reciprocally,
$H_{a,b}$ is a **unfolding** of $\widetilde{Q}_a(x) = (1 - ax^2, 0)$

Good unfoldings of the limit family

Let $\Psi_a(x,y) = (\varphi_a(x), 0)$, where φ_a is a family of smooth unimodal maps verifying the six previous conditions (d.M, v.S).
Let $F_{a,b}(x,y) = \Psi_a(x,y) + \Delta_{a,b}(x,y)$ defined on a domain $\widetilde{D}$, and

$$DF_{a,b} = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

Definition

$F_{a,b}(x,y)$ is said a *good unfolding of the limit family* $\Psi_a(x,y)$ iff

$$\|\Delta\|_{C^3(\widetilde{D})} \leq K\sqrt{b}$$

and, in adition, it holds the following techniques assumptions:

- $|A| \leq K, |B| \leq K\sqrt{b}, |C| \leq K\sqrt{b}, |D| \leq Kb, |detDF_{a,b}| \leq Kb.$
- $|D_{(a,x,y)}A| \leq K, |D_{(a,x,y)}B| \leq K\sqrt{b}, |D_{(a,x,y)}C| \leq K\sqrt{b}, |D_{(a,x,y)}D| \leq Kb.$
- $|D^2_{(a,x,y)}A| \leq K, |D^2_{(a,x,y)}B| \leq K\sqrt{b}, |D^2_{(a,x,y)}C| \leq K\sqrt{b}, |D^2_{(a,x,y)}D| \leq Kb.$

Hénon-like family

The above conditions are assumed to get that the unfolding inherits the properties of the unimodal family: expansivity and transitivity. This occurs in the case of Hénon-like family.

Definition

A good unfolding $F_{a,b}(x,y)$ of the quadratic family

$$\Psi_a(x,y) = (1 - ax^2, 0)$$

is called a Hénon-like family.

Existence of persistent strange attractors in Hénon-like family was proved by Mora and Viana. The same is proved in the book with Pumariño for a good unfolding of the unimodal family

$$\Psi_a(x,y) = (\lambda^{-1} \ln a + x + \lambda^{-1} \ln \cos x, 0)$$

in order to get persistent strange attractors in three-dimensional flows.

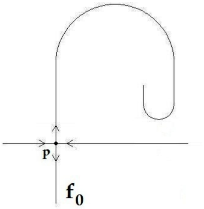
In fact, Mora and Viana proved that Hénon-like families can be defined in a neighborhood of a tangent homoclinic point when a generic homoclinic bifurcation takes place. Therefore, they proved the following conjecture of Jacob Palis.

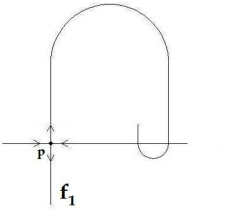
Conjecture

Generic one-parameter families of surface diffeomorphisms unfolding a homoclinic tangency exhibit strange attractors or repeller in a persistent way in the measure theoretic sense.

Let $\{f_\mu : M \to M\}_{\mu \in [0,1]}$ be a family of diffeomorphisms with a saddle fixed point p_μ and such that

- f_0 has no homoclinic orbit.
- f_1 has a transversal homoclinic orbit.
- $\mu \in [0,1] \to f_\mu$ is continuous.





Unfoldings of homoclinic tangencies

Let $\{f_\mu : M \to M\}_{\mu\in[0,1]}$ be a family of diffeomorphisms with a saddle fixed point p_μ and such that

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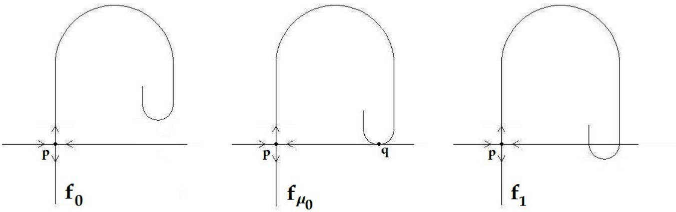
- f_1 has a transversal homoclinic orbit.
- $\mu \in [0,1] \to f_\mu$ is continuous.

There exists a value μ_0 of μ such that f_{μ_0} has a **tangent homoclinic point** q



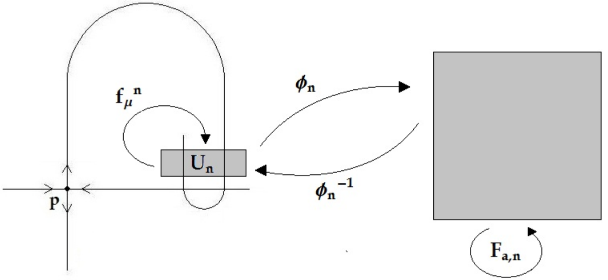
Hénon-like family and homoclinic tangencies

Hénon-like families for generic unfoldings of a homoclinic tangency are defined by means of:

- A change of parameter $a = a(\mu)$ in a neighborhood of $\mu = \mu_0$.
- A variables change in a neighborhood of the homoclinic point q .

Then

$$F_{a,n} = \Phi_n \circ f_\mu^n \circ \Phi_n^{-1} \text{ such that } lim_{n\to\infty} F_{a,n} = (1 - ax^2, 0).$$



Hénon-like family and homoclinic tangencies in dimension $n > 2$

The previous renormalization can also work in dimension $n > 2$ in order to yield one-dimensional strange attractors. For instance, if $f_\mu : M \to M$ unfolds a homoclinic tangency associated to a sectionally dissipative periodic point p.

Definition

The periodic point p is said sectionally dissipative if its eigenvalues $\lambda_1, \lambda_2, ..., \lambda_n$ satisfy

$$|\lambda_1| < |\lambda_2| < ... < |\lambda_{n-1}| < 1 < |\lambda_n|$$

and

$$|\lambda_{n-1}\lambda_n| < 1$$

Under this assumption the family of limit return maps is given by

$$F_a(x_1, x_2, ..., x_n) = (1 - ax_1^2, 0, ..., 0)$$

Non-hyperbolic dynamics in dimension two

References

- **Palis, J.; Viana, M.** *High dimension diffeomorphisms displaying infinitely many sinks*. Ann. Math., 140, 1994.
- **Romero, N.** *Persistence of homoclinic tangencies in higher dimensions*. Ergod. Th. Dyn. Sys. 15, 1995.

The gates to pass from hyperbolic to non-hyperbolic dynamics are the homoclinic bifurcations. These are equivalent to creation or destruction of horseshoes. This equivalence and the notion of limit map allow to explain the following non-hyperbolic phenomena:

- Infinitely many sinks (Newhouse phenomenon).
- Existence of persistent (probable) strange attractors.
- Infinitely many non-persistent strange attractors.

New references

- **Colli, E.** *Infinitely many coexisting strange attractors*. Ann. Inst. H. Poincaré Anal. Non Lináire, 15, 1998.
- **Newhouse, S.** *Diffeomorphisms with infitely many sinks*. Topology 9, 1974.

Non-hyperbolic dynamics in dimension $n \geq 3$

Homoclinic tangencies can now yield non-hyperbolic persistent **two-dimensional** strange attractors, whenever the unstable manifold has dimension two.

In addition to the homoclinic tangencies, the existence of heterodimensional cycles is a new mechanism implying robust non-hyperbolic dynamics.

Associated to heterodimensional cycles can be introduced the notion of blender (something like a generalization of the horseshoe) and an iterated functions system (IFS).

Proving the existence of two-dimensional strange attractors and researching the dynamics of IFS look like two interesting problems in non-hyperbolic dynamics.

We will just talk about the existence of two-dimensional strange attractors. But, **it seems natural to wonder if strange attractors and IFS may be closely related.**

Part III

Two-dimensional strange attractors

Approach to the problem

Challenge

To prove the existence of persistent two-dimensional strange attractors for diffeomorphisms on $\mathbb{R}^3$ in a generic context.

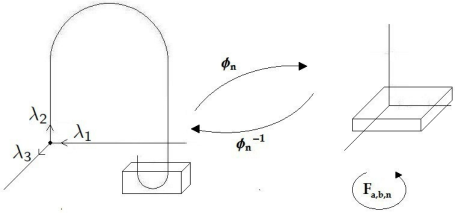
Assumptions

- $\{f_{a,b}\}_{a,b}$ is a two-parametric family of diffeomorphisms in $\mathbb{R}^3$
- Existence of a dissipative saddle fixed point, but no sectionally dissipative. The eigenvalues satisfy $0 < |\lambda_1| < 1 < |\lambda_2| < |\lambda_3|$
- Existence of generalized homoclinic tangency, which is generically unfolded (codimension two)

References

- **Tatjer, J.C.** *Three-dimensional dissipative diffeomorphisms with homoclinic tangencies*. Ergod. Th. Dyn. Sys. 21, 2001.
- **Gonchenko, S.V.; Gonchenko, V.S., Tatjer, J.C.** Regular and Chaotic Dynamics 12 (2007)

Limit family for the case 3D dissipative (non s. d.)



$$|\lambda_1| < 1 < |\lambda_2| < |\lambda_3|$$
$$|\lambda_1 \lambda_2 \lambda_3| < 1$$

By means of an adequate renormalization the following limit family is obtained

$$F_{a,b,n}(x,y,z) \underset{n\to\infty}{\longrightarrow} F_{a,b}(x,y,z) = (z, a+by+z^2, y)$$

Two-dimensional limit family

For each $(a, b) \in \mathbb{R}^2$ every point in $\mathbb{R}^3$ falls by one iteration of $F_{a,b}$ into the surface

$$C_{a,b} = \{(x, y, z) : y = a + bz + x^2\}$$

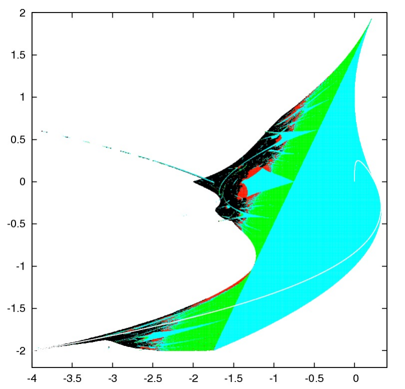
On the other hand it is not difficult to see that $F_{a,b}$ restricted to $C_{a,b}$ is conjugate to the family of endomorphisms defined on $\mathbb{R}^2$ by

$$T_{a,b}(x, y) = (a + y^2, x + by)$$

References

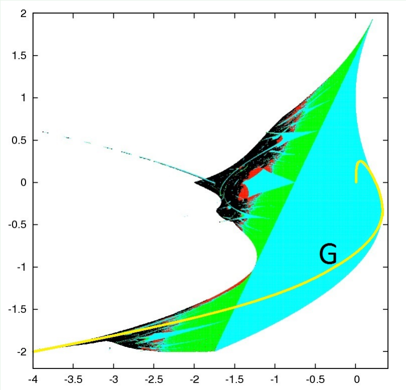
- **Tatjer, J. C.** *Three-dimensional dissipative diffeomorphisms with homoclinic tangencies*. Ergodic Theory and Dynamical Systems, 21 (2001), 249-302.
- **Pumariño, A. and Tajter, J. C.** *Dynamics near homoclinic bifurcations of three-dimensional dissipative diffeomorphisms* Nonlinearity, 19, 2006.
- **Pumariño, A. and Tatjer, J. C.** *Attractors for return maps near homoclinic tangencies of three-dimensional dissipative diffeomorphisms*. Discrete and Continuous Dynamical Systems, series B, vol 8, n^o 4, 2007.

Numerical results for parameters with invariant domains



- Blue: attractor periodic point.
- Green: attractor formed by a junction of closed curves.
- Red: one-dimensional strange attractor.
- Black: two-dimensional strange attractor.

Numerical results for parameters with invariant domains



- Blue: attractor periodic point.
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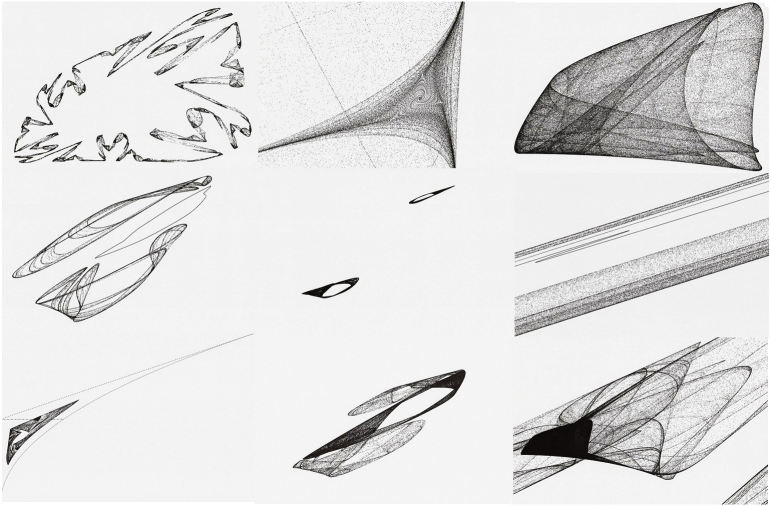
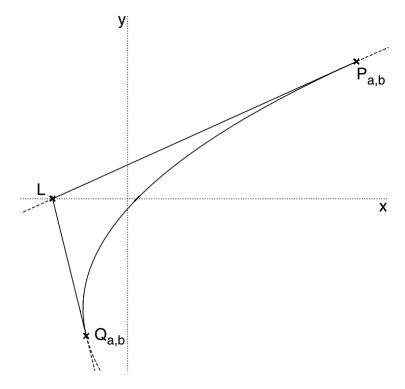
$$G = \{(a(s), b(s)) = \left(-\frac{s^3}{4}(s^3 - 2s^2 + 2s - 2), -s^2 + s\right) \; : \; s \in [0, 2]\}$$

Possible strange attractors



Invariant domain along the curve G

Invariant domain of $T_{a(s),b(s)}$: $\mathcal{D}_s$



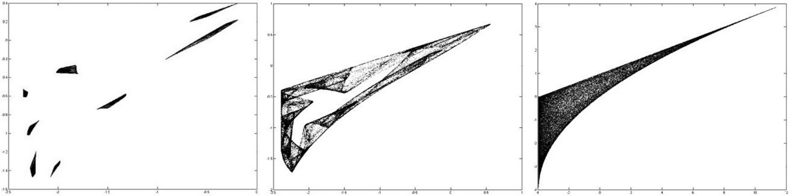
- Si $s < 2$,

$$T_{a(s),b(s)}(\mathcal{D}_s) \subset \mathcal{D}_s$$

- Si $s = 2, a = -4, b = -2$

$$T_{-4,-2}(\mathcal{D}_2) = \mathcal{D}_2$$

Possible strange attractors along the curve G



The special case: $(a, b) = (-4, -2)$

Let us define $\mathcal{T} = \mathcal{T}_0 \cup \mathcal{T}_1$ with

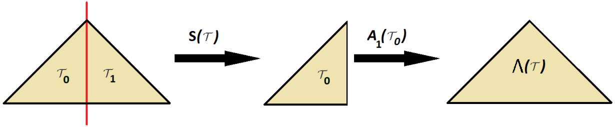
$$\mathcal{T}_0 = \{(x, y) : \; 0 \leq x \leq 1, \; 0 \leq y \leq x\}$$
$$\mathcal{T}_1 = \{(x, y) : \; 1 \leq x \leq 2, \; 0 \leq y \leq 2 - x\}$$

Proposition

The map $T_{-4,-2}|_{\mathcal{D}_2}$ is conjugate to $\Lambda_1 = A_1 \circ \mathcal{S}$, where

$$\mathcal{S}(x, y) = \begin{cases} (x, y) & si \quad (x, y) \in \mathcal{T}_0 \\ (2 - x, y) & si \quad (x, y) \in \mathcal{T}_1 \end{cases}$$

$$A_1 = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}.$$

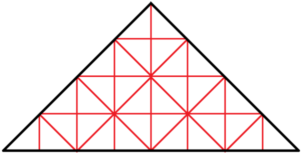


Theorem

The map Λ_1 (and so $T_{-4,-2}$) verify:

1. They are conjugate to the shift with two simbols.
2. They have a dense orbit in the invariant domain with two positive Lyapunov exponents.
3. They have a unique absolutely continua invariant (ergodic) measure.

Λ_1 is called **two-dimensional tent map**.



A. Pumariño and J. C. Tajter,
Dynamics near homoclinic bifurcations of three-dimensional dissipative diffeomorphisms Nonlinearity, 19, 2006.

A good example of two-dimensional strange attractor?

As far as we know, Λ_1 (and so $T_{-4,-2}$) yield the first example of a two-dimensional strange attractor for one endomorphism T such that none n-composition T^n is C^1-conjugate to a triangular map. Nevertheless, we would like $T_{-4,-2}$ to be a diffeomorphism. Moreover:

We would like to find such attractor for an unfolding of $T_{-4,-2}$ that it was a return map on a neighborhood of a generalized homoclinic tangency

However:

To prove the existence of strange attractors from limit family to the corresponding family of diffeomorphisms is a very difficult task. **It was seen even in the one-dimensional case**

Part IV

Two-dimensional strange attractors for limit families: Expanding baker maps (EBM)

Two-dimensional strange attractors for limit families: Expanding baker maps

Existence of persistent one-dimensional strange attractors for

$$f_a(x) = 1 - ax^2$$

was proved by means of a carefully process of excluding parameters, in order to prevent that the critical orbit returns too close to the critical point.

A similar process could be proposed for the limit family

$$T_{a,b}(x, y) = (a + y^2, x + by) \quad .$$

However, in this case we have a critical curve instead of a critical point.

As a first approach we will work with piecewise linear maps: **Expanding baker maps**.

Two-dimensional strange attractors for limit families: Expanding baker maps

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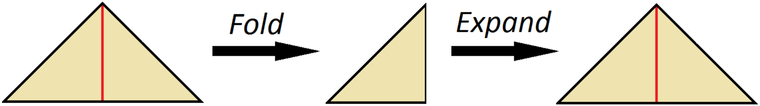


History in dimension one

To compare maps with sensitivity to simple maps (piecewise affine maps) has a long history in dimension one: Section 8 of Chapter II in the book by W. de Melo and S. van Strien.

For example,

If $f_a(x) = 1 - ax^2$ **has no stable periodic points and no restrictive central point, then there exists** a' **such that** $\lambda_{a'}(x) = 1 - a'|x|$ **and** f_a **are conjugate.**

References

- **Collet P.; Eckmann, J. P.** *Iterated maps of the interval as dynamical systems.* Birkhauser, 1980.

Definitions

Fold

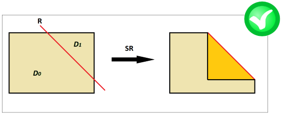
Let D be a polygonal domain, $P \in D$ y R a straight line that divides D into two subsets D_0 y D_1 (w.l.g. $P \in D_0$). We define the **fold** of D by R as

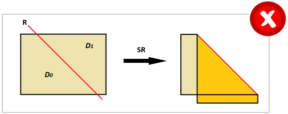
$$\mathcal{S}_R(x,y) = \begin{cases} (x,y) & \text{if} \quad (x,y) \in D_0 \\ (\overline{x},\overline{y}) & \text{if} \quad (x,y) \in D_1 \end{cases}$$

where $(\overline{x},\overline{y})$ is the symmetric of (x,y) with respect to R.

Good fold

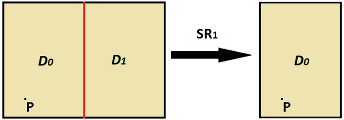
We say that the fold $\mathcal{S}_R$ is **good** if $\mathcal{S}_R(D) = D_0$.





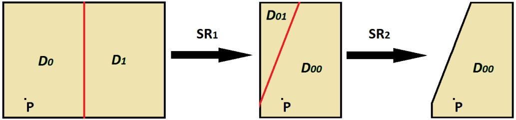
Sequence of folds

- Let $\mathcal{S}_{R_1}$ be a good fold defined in the domain D by means of a straight R_1 in order to obtain D_0 .



Sequence of folds

- Let $\mathcal{S}_{R_1}$ be a good fold defined in the domain D by means of a straight R_1 in order to obtain D_0 .
- Let $\mathcal{S}_{R_2}$ be a good fold defined in the domain D_0 by means of R_2. in order to obtain D_{00}.



Sequence of folds

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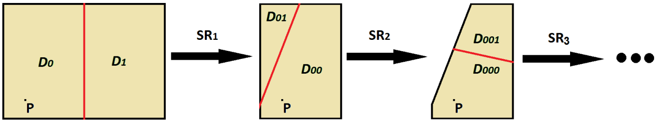
- Let $\mathcal{S}_{R_2}$ be a good fold defined in the domain D_0 by means of R_2. in order to obtain D_{00}.
- Let us iterate the process for $R_3 \ldots R_n$



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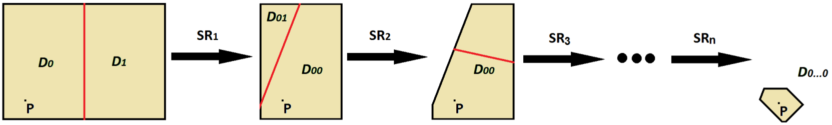
After n good folds we obtain $D_{0\dots0} \subset D$ with $P \in D_{0\dots0}$.



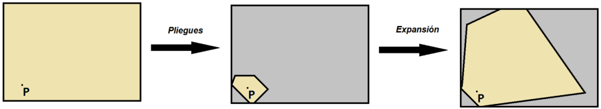
Definition of EBM

Let D be a polygonal domain, $P \in D$ and $\mathcal{S}_{R_1} \dots \mathcal{S}_{R_n}$ a sequence of good folds. Let $T_P(Q) = Q - P$ and $M \in \mathcal{M}_{2x2}$ such that

$$T_p^{-1} M T_p(D_{0 \overset{n}{\dots} 0}) \subset D.$$

We define the **expanding baker map** Λ as the map given by

$$\Lambda = T_P^{-1} \circ M \circ T_P \circ S_{R_n} \circ \dots \circ S_{R_1}$$



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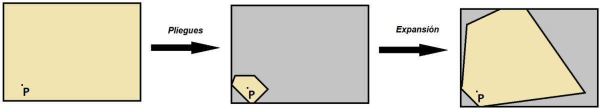
Two-dimensional tent map

$D = \mathcal{T},\ P = (0,0),\ n = 1,\ R_1 = \mathcal{C} = \{(x,y):\ x = 1\},\ M = A_t$



Expanding Baker Maps associated to $T_{a(s),b(s)}$

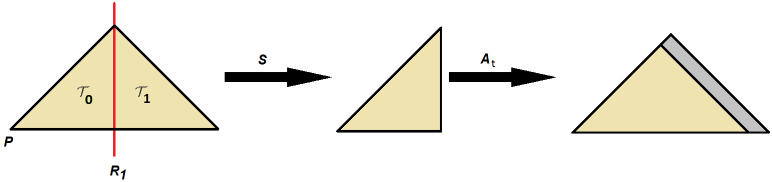
In order to reproduce the dynamics observed for $T_{a(s),b(s)}$ we introduce the family of EBM $\{\Lambda_t\}_{0\leq t\leq 1}$ given by

$$\Lambda_t = A_t \circ \mathcal{S} : \mathcal{T} \to \mathcal{T}$$

where

$$\mathcal{S}(x,y) = \begin{cases} (x,y) & \textit{if} \quad (x,y) \in \mathcal{T}_0 \\ (2-x,y) & \textit{if} \quad (x,y) \in \mathcal{T}_1 \end{cases}$$

$$A_t = \begin{pmatrix} t & t \\ t & -t \end{pmatrix}.$$



The choice of $\{\Lambda_t\}_{0\leq t\leq 1}$ was motivated and its dynamics numerically studies in

- **Pumariño, A.; Rodríguez, J.A.; Tatjer, J.C.; Vigil, E.** *Expanding baker maps as models for the dynamics emerging from 3D-homoclinic bifurcations*. Discrete and Cont. Dynamical Sys, series B, vol 19 n. 2 (2014).

- **Pumariño, A.; Rodríguez, J.A.; Tatjer, J.C.; Vigil, E.** *Piecewise linear bidimensional maps as models of return maps for 3D-diffeomorphisms*. Proceeding of International Congress DS100 Year After Poincaré. Springer (2013)

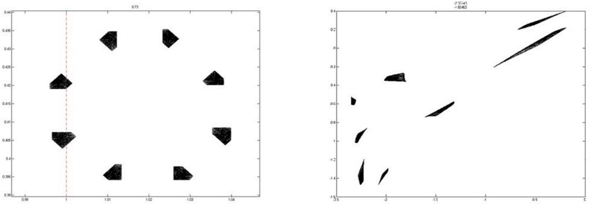
Dinamics of Λ_t

Let $t^* = \frac{1}{\sqrt{2}}$. The dynamics of Λ_t is simple for $t \leq t^*$

- If $t < t^*$, there exists a unique fixed point (in the origin) which is a global attractor.
- For $t = t^*$ every point of $\Lambda^2_{t^*}(T)$ is fixed by $\Lambda^2_{t^*}$.
- The interesting dynamics takes place when $t > t^*$: a new fixed point P_t appear and the map Λ_t becomes expansive. The origin is a repelling nodus while P_t emerge as a repelling focus. Then non-trivial attractors (connected,non simply connected, and non-connected) arise.

Non-connected attractors

In the case $\frac{1}{\sqrt{2}} < t < \frac{1}{\sqrt[5]{4}}$, the attractor consists of eight pieces.



- On the left, the attractor for Λ_t with $t = 0.73$.
- On the right, the attractor for $T_{a(s),b(s)}$ with $s = 1.8909$.

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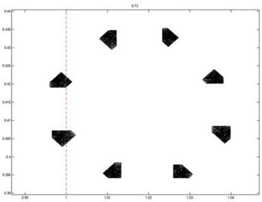

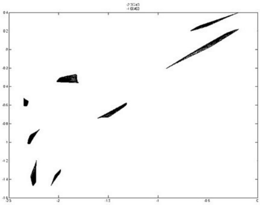

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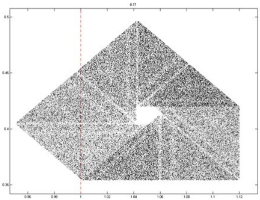
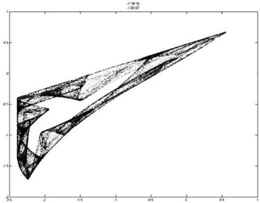
We will call this type of attractors *Fairy Cakes Attractor*

Non simply connected attractors

In the case $\frac{1}{\sqrt[5]{4}} < t < \frac{1}{\sqrt[3]{2}}$, the attractor consists of a single piece with a hole.





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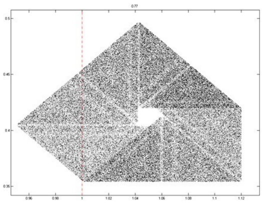

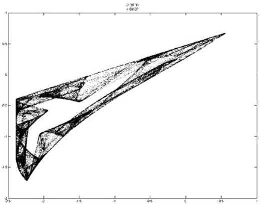

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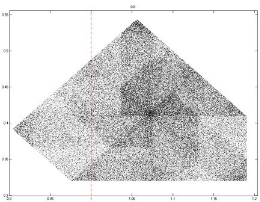
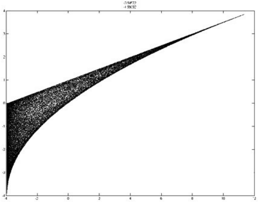
This type of attractor will be called *Bread Rolls Attractor*

In the case $\frac{1}{\sqrt[3]{2}} < t < 1$ the attractor consists of a unique piece without hole.

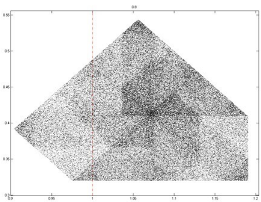


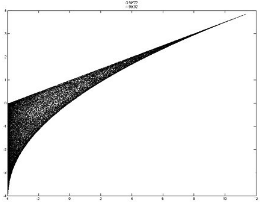


- On the left, the attractor for Λ_t with $t = 0.8$.
- A la right, the attractor para $T_{a(s),b(s)}$ with $s = 1.99$.

Connected attractor

In the case $\frac{1}{\sqrt[3]{2}} < t < 1$ the attractor consists of a unique piece without hole.





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This attractor can be named
Country Bread Attractor

Part V

Are these attractors really strange?

Analytical proof

Theorem

There exists an interval of parameters $\mathcal{I} \subset [\frac{1}{\sqrt{2}}, 1]$ such that for every $t \in \mathcal{I}$ the map Λ_t displays a two-dimensional strange attractor $\mathcal{R}_t \subset \mathcal{T}$. Moreover, $\mathcal{R}_t$ supports a unique absolutely continuous and ergodic invariant measure.

A. Pumariño, J. A. Rodriguez, J. C. Tatjer and E. Vigil, *Chaotic dynamics for 2-D tent maps.* Submitted for publication in Nonlinearity

Comments for the proof

Remarks

- Actually, $\mathcal{I} \subset [\frac{1}{\sqrt[3]{2}}, 1]$. Hence $\mathcal{R}_t$ will be a pentagon according to the previous graphic (Country Bread Attractor)
- Both Lyapounov exponents are: $\log(\sqrt{2}t)$ $(t > 1/\sqrt{2})$
- The main difficulty is to prove that $\mathcal{R}_t$ is **transitive**.
- The existence of a unique absolutely continuous and ergodic invariant measure follows from the transitivity and the results by J. Buzzi, and B. Saussol.

References

- **Buzzi, J.** *Absolutely continuous invariant measures for generic multidimensional piecewise affine expanding maps*. Inter. Jour. Bif. and Chaos 9, 1981.
- **Saussol, B.** *Absolutely continuous invariant measures for multidimensional expanding maps*. Israel Journal of Mathematics, 116, 2000.

Transitivity of $\mathcal{R}_t$

Definition

$\mathcal{R}_t$ is transitive if $\Lambda_{t|\mathcal{R}_t}$ is topologically transitive: for every pair of open sets $U, V \subset \mathcal{R}_t$ there exists a natural number n such that

$$\Lambda_t^n(U) \cap V \neq \emptyset.$$

Alternative

Density of the pre-fixed points:

$$O^-(P_t) = \{q : \ \exists n \in \mathbb{N} \text{ tal que } \Lambda_t^n(q) = P_t\}.$$

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This property is stronger than the transitivity of $\mathcal{R}_t$

Results for the proof

Proposition

Let $\mathcal{B} = B(q, \epsilon) \subset \mathcal{R}_t$ a disk such that $\mathcal{B} \cap \mathcal{C} \neq \emptyset$. Then at least one of the following statements holds:

i) There exists a pre-fixed point in $\mathcal{B}$

ii) *(Expansitivity)* There exists a disk $\mathcal{B}_1 \subset \mathcal{B}$ and a natural number n_1 such that $\Lambda_t^{n_1}(\mathcal{B}_1)$ is a disk of radio $\epsilon_1 > \epsilon$.

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Corollary 1: Density of the pre-fixed points

For every open set $U \subset \mathcal{R}_t$ there exists a point $x \in U$ such that $\Lambda_t^n(x) = P_t$ for some $n \in \mathbb{N}$.

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Corollary 2: Transitivity

For every open set $U \subset \mathcal{R}_t$ there exists a natural number $n \in \mathbb{N}$ such that $\Lambda_t^n(U) = \mathcal{R}_t$.

Sketch of the proof: staging

Let us return to the family *EBM* $\{\Lambda_t\}_{0\leq t\leq 1}$ given by

$$\Lambda_t = A_t \circ \mathcal{S} : \mathcal{T} \to \mathcal{T}$$

where

$$\mathcal{S}(x,y) = \begin{cases} (x,y) & \textit{if} \quad (x,y) \in \mathcal{T}_0 \\ (2-x,y) & \textit{if} \quad (x,y) \in \mathcal{T}_1 \end{cases}$$

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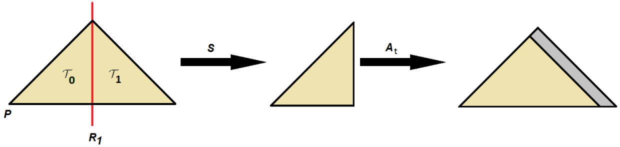
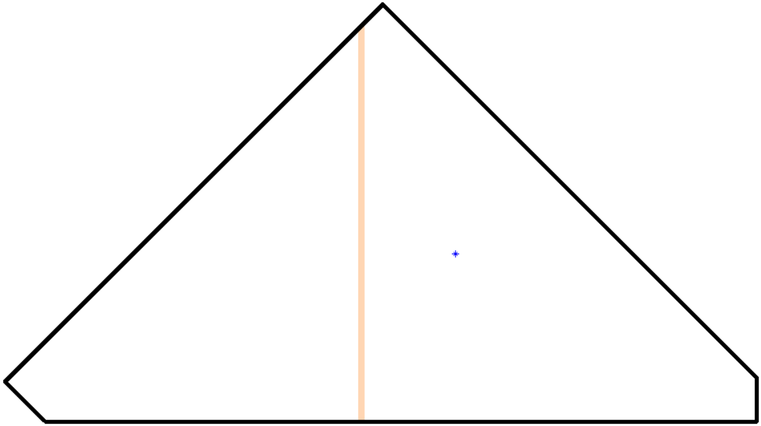


Sketch of the proof: partition and symmetries

$\mathcal{C}_0 = \mathcal{C} \cap \mathcal{R}_t$



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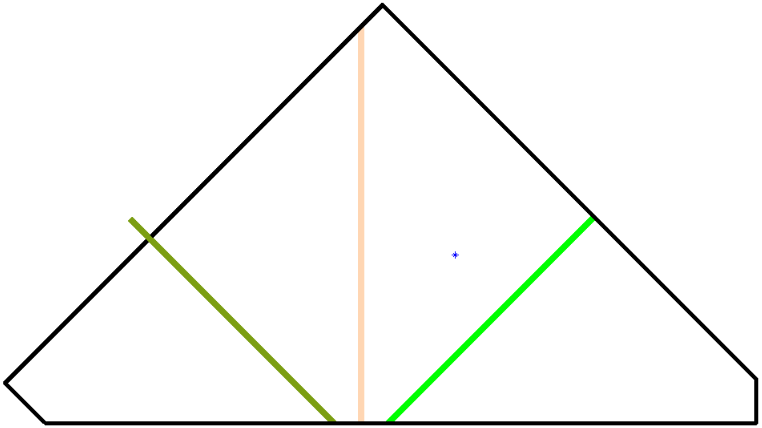

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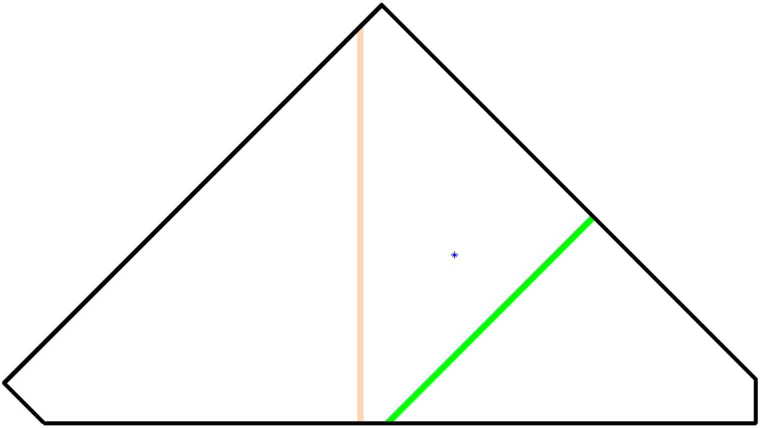

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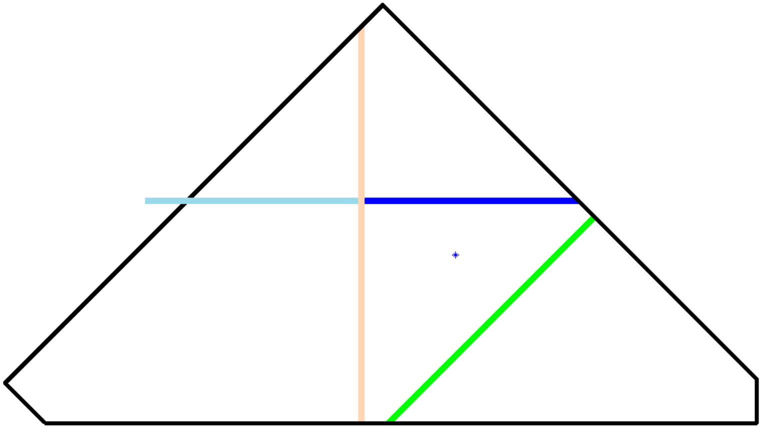

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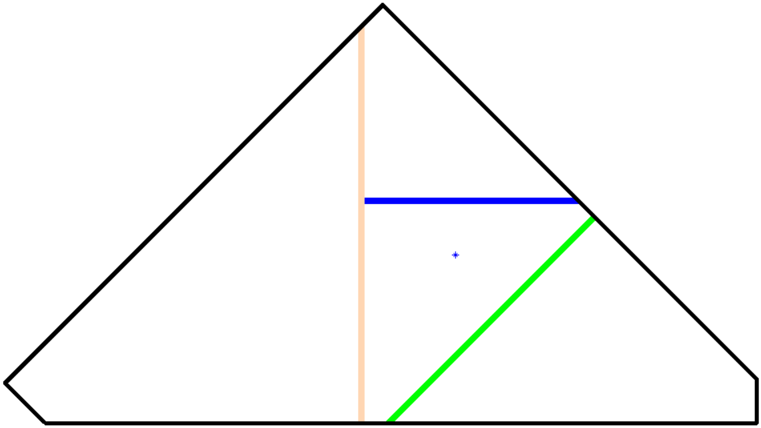

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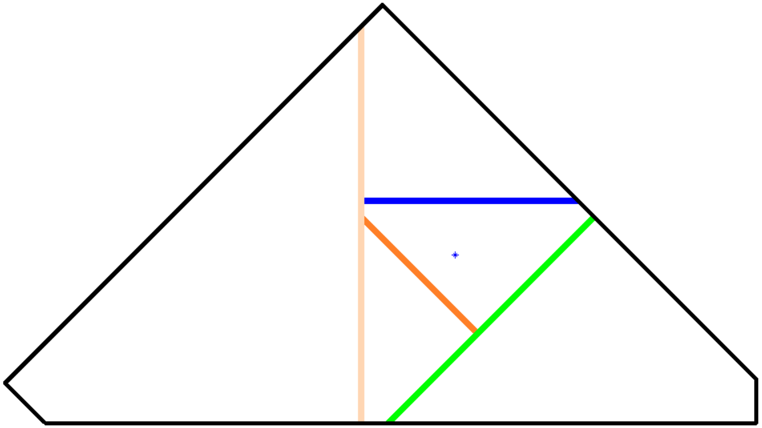

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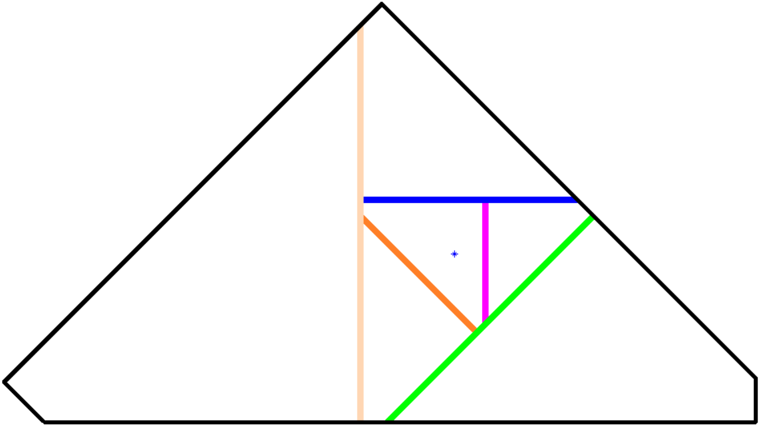

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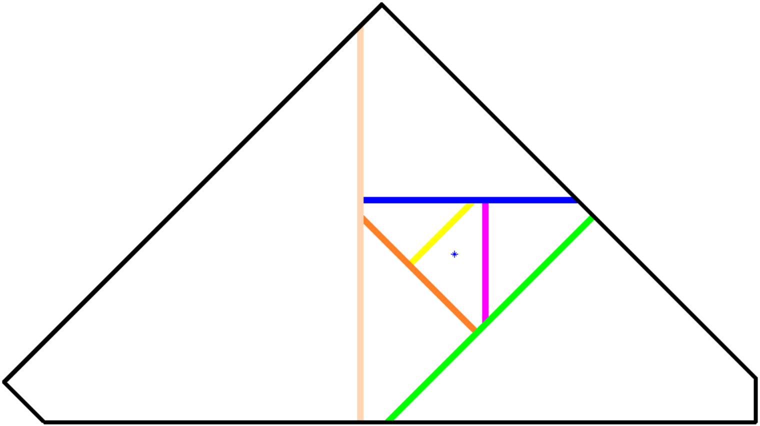

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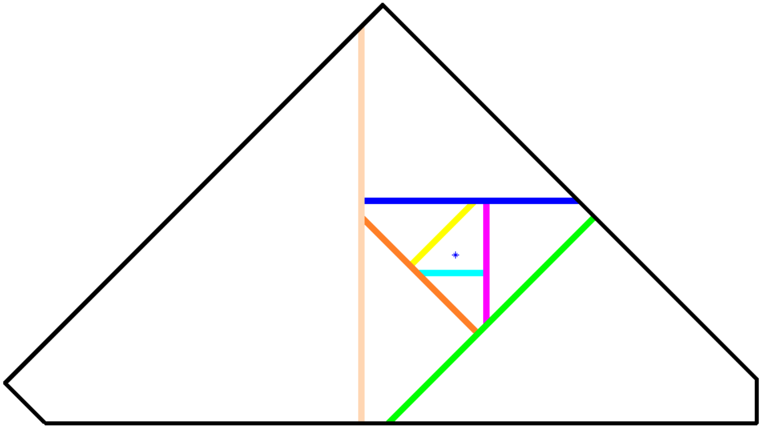

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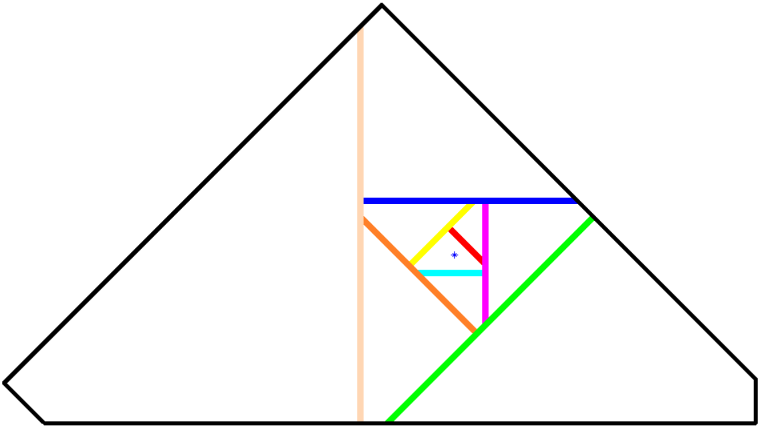

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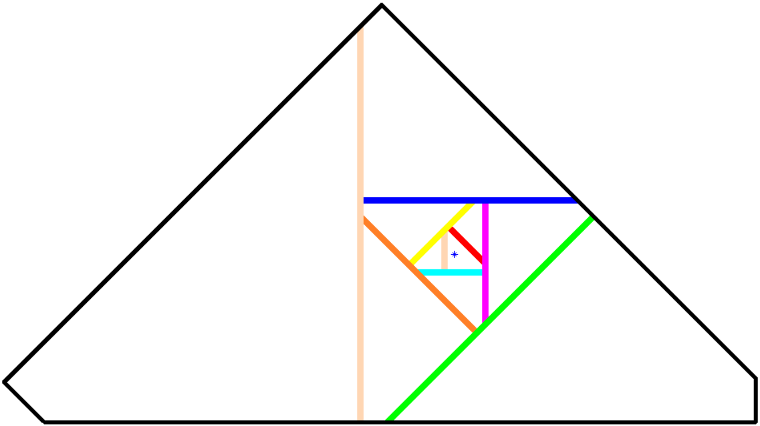
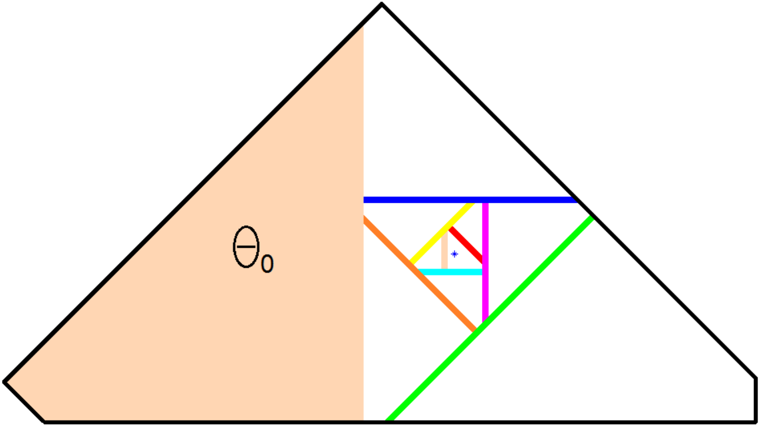


Sketch of the proof: partition and symmetries

$\Theta_0 = \mathcal{T}_0 \cap \mathcal{R}_t$



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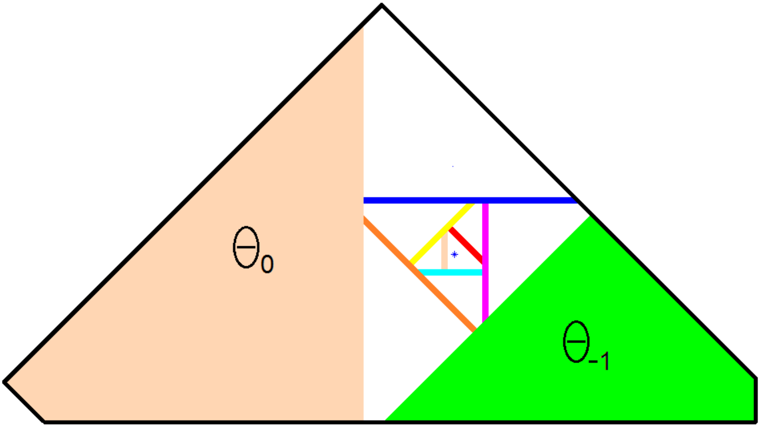

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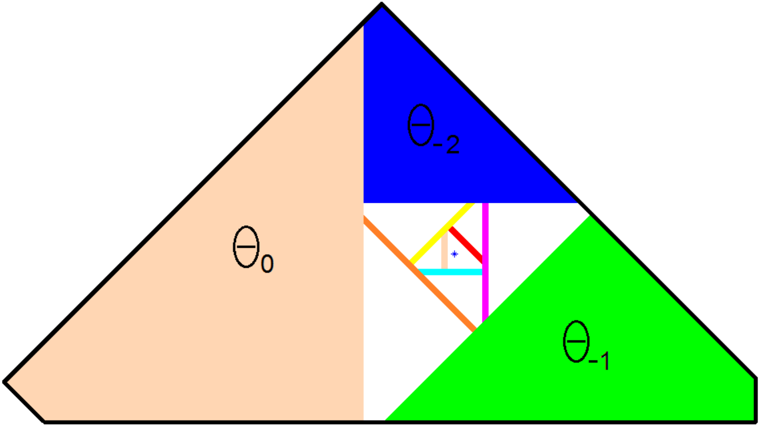

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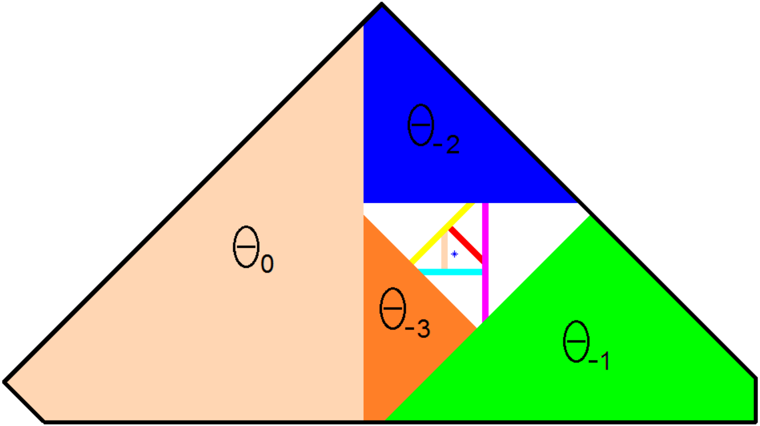

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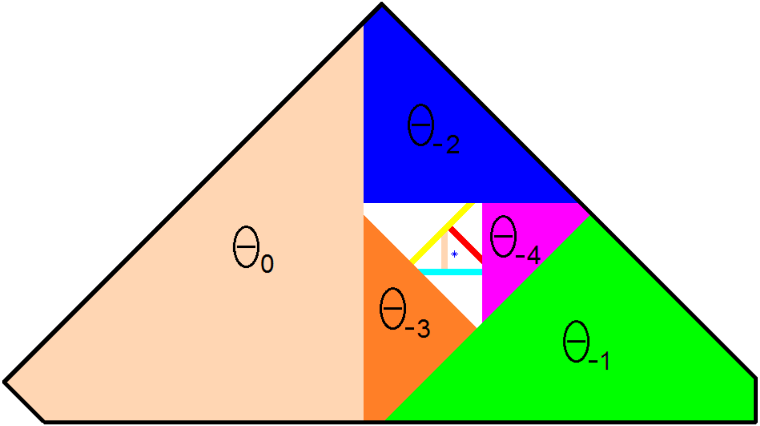

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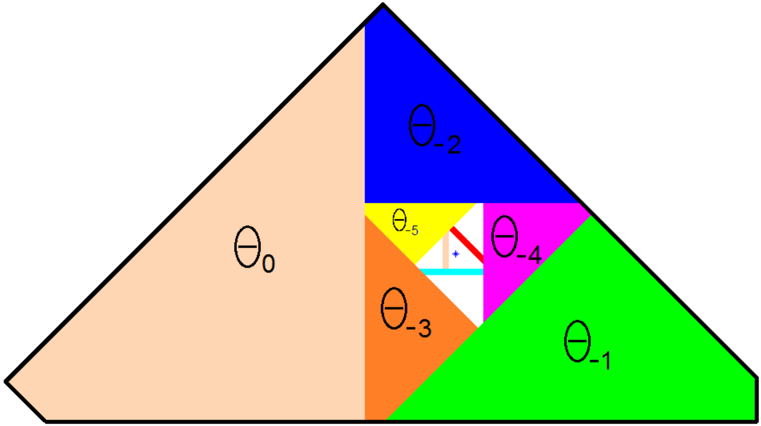

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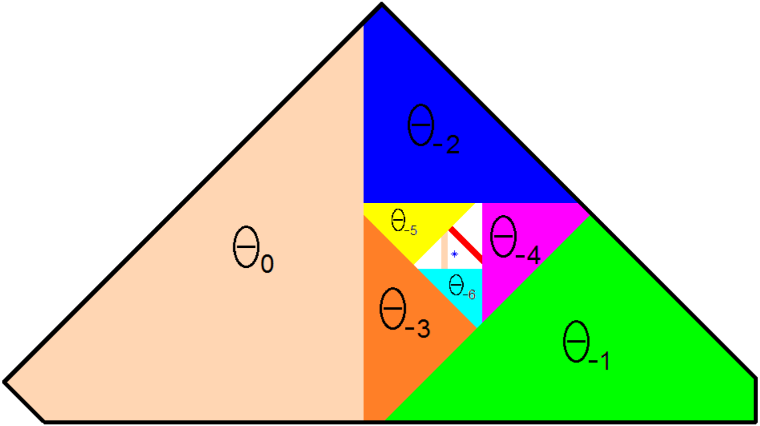

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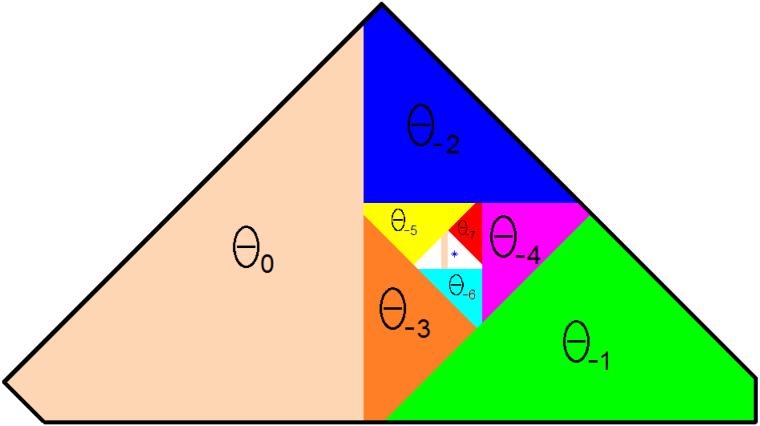

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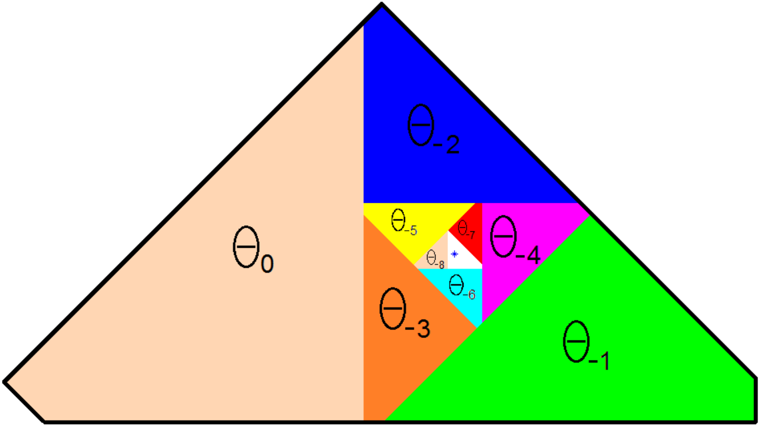


Sketch of the proof: partition and symmetries

Pre-critical lines

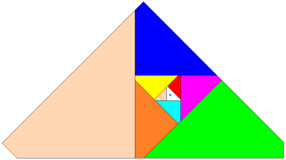
$$\mathcal{C}_0 = \mathcal{C} \cap \mathcal{R}_t$$

$$\mathcal{C}_{-k} = \Lambda_t^{-1}(\mathcal{C}_{-k+1}) \cap \mathcal{T}_1, \ \ k \in \mathbb{N}$$

Partition

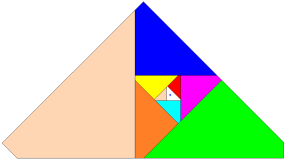
$$\Theta_0 = \mathcal{R}_t \cap \mathcal{T}_0$$

$$\Theta_{-k} = \Lambda_t^{-1}(\Theta_{-k+1}) \cap \mathcal{T}_1, \ \ k \in \mathbb{N}$$



Propierties

1. The union of all pre-critical lines is dense in $\mathcal{R}_t$.



Sketch of the proof: partition and symmetries

Propierties

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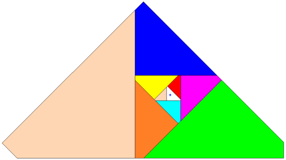
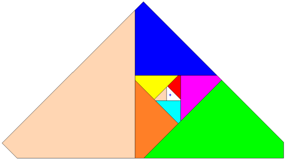
2. $\bigcup_{k\geq 0} \Theta_{-k} = \mathcal{R}_t \setminus \{P_t\}$



Sketch of the proof: partition and symmetries

Propierties

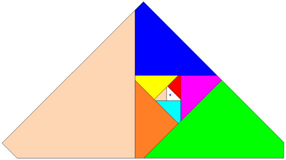
1. The union of all pre-critical lines is dense in $\mathcal{R}_t$.
2. $\bigcup_{k \geq 0} \Theta_{-k} = \mathcal{R}_t \setminus \{P_t\}$
3. $int(\Theta_{-i}) \cap int(\Theta_{-j}) = \emptyset$ for each $i \neq j$.



Sketch of the proof: partition and symmetries

Propierties

1. The union of all pre-critical lines is dense in $\mathcal{R}_t$.
2. $\bigcup_{k \geq 0} \Theta_{-k} = \mathcal{R}_t \setminus \{P_t\}$
3. $int(\Theta_{-i}) \cap int(\Theta_{-j}) = \emptyset$ for each $i \neq j$.
4. For every $k \geq 0$ there exists n_k such that $\Lambda_t^{n_k}(\Theta_{-k}) = \mathcal{R}_t$.



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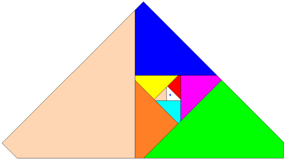
5. Let $x \in \Theta_{-k}$ and $\overline{x}$ its symmetric with respect to $\mathcal{C}_{-k}$. Then $\Lambda_t^{k+1}(x) = \Lambda_t^{k+1}(\overline{x})$



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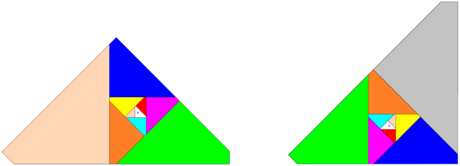
6. For every $n \geq 0$ the set $\bigcup_{k\geq n} \Theta_{-k}$ is similar to $\mathcal{R}_t$.



Sketch of the proof: Proposition

Proposition

Let $\mathcal{B} = B(q, \epsilon) \subset \mathcal{R}_t$ be a disk such that $\mathcal{B} \cap \mathcal{C} \neq \emptyset$. Then at least one of the following statements holds:

i) There exists a pre-fixed point in $\mathcal{B}$

ii) *(Expansivity)* There exists a disk $\mathcal{B}_1 \subset \mathcal{B}$ and a natural number n_1 such that $\Lambda_t^{n_1}(\mathcal{B}_1)$ is a disk of radio $\epsilon_1 > \epsilon$.

Proof

Let k be such that $q \in \Theta_{-k}$.

- If $k \notin \{0, 2, 3, 5\}$ then $\mathcal{B}$ contains a pre-fixed point
- If $k \in \{3, 5\}$ then either $\mathcal{B}$ contains a pre-fixed point or the expansivity condition ii) holds.
- If $k \in \{0, 2\}$ the proof reduces to the previous cases by taking the symmetric disk of $\mathcal{B}$ with respect to $\mathcal{C}_{-k}$.

Sketch of the proof: corollary 1

Corollary1

For every open set $U \subset \mathcal{R}_t$ there exists a point $x \in U$ such that $\Lambda_t^n(x) = P_t$ for some $n \in \mathbb{N}$.

Proof

Sketch of the proof: corollary 1

Corollary1

For every open set $U \subset \mathcal{R}_t$ there exists a point $x \in U$ such that $\Lambda_t^n(x) = P_t$ for some $n \in \mathbb{N}$.

Proof

Let n_0 be the first natural number such that $\Lambda_t^{n_0}(U) \cap \mathcal{C} \neq \emptyset$. Then

$\mathcal{B}_0 \subset \Lambda_t^{n_0}(U)$

Sketch of the proof: corollary 1

Corollary1

For every open set $U \subset \mathcal{R}_t$ there exists a point $x \in U$ such that $\Lambda_t^n(x) = P_t$ for some $n \in \mathbb{N}$.

Proof

Let n_0 be the first natural number such that $\Lambda_t^{n_0}(U) \cap \mathcal{C} \neq \emptyset$. Then

$$\mathcal{B}_0 \subset \Lambda_t^{n_0}(U) \stackrel{Prop.ii)}{\longrightarrow} \mathcal{B}_1 \subset \Lambda_t^{n_1}(\mathcal{B}_0)$$

Sketch of the proof: corollary 1

Corollary1

For every open set $U \subset \mathcal{R}_t$ there exists a point $x \in U$ such that $\Lambda_t^n(x) = P_t$ for some $n \in \mathbb{N}$.

Proof

Let n_0 be the first natural number such that $\Lambda_t^{n_0}(U) \cap \mathcal{C} \neq \emptyset$. Then

$$\mathcal{B}_0 \subset \Lambda_t^{n_0}(U) \xrightarrow{Prop.ii)} \mathcal{B}_1 \subset \Lambda_t^{n_1}(\mathcal{B}_0) \xrightarrow{Prop.ii)} \mathcal{B}_2 \subset \Lambda_t^{n_2}(\mathcal{B}_1)$$

Sketch of the proof: corollary 1

Corollary1

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Let n_0 be the first natural number such that $\Lambda_t^{n_0}(U) \cap \mathcal{C} \neq \emptyset$. Then

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Sketch of the proof: corollary 1

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For every open set $U \subset \mathcal{R}_t$ there exists a point $x \in U$ such that $\Lambda_t^n(x) = P_t$ for some $n \in \mathbb{N}$.

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namely, we obtain a sequence of disks $\{\mathcal{B}_n\}$ with radius $\mu_0 < \mu_1 < \mu_2 < \dots$. Since this process has to be finite, there exists a natural number m such that $\mathcal{B}_m$ contains a pre-fixed point.

Sketch of the proof: corollary 2

Corollary 2

For every open set $U \subset \mathcal{R}_t$ there exists a natural number $n \in \mathbb{N}$ such that $\Lambda_t^n(U) = \mathcal{R}_t$.

Demostración

Sketch of the proof: corollary 2

Corollary 2

For every open set $U \subset \mathcal{R}_t$ there exists a natural number $n \in \mathbb{N}$ such that $\Lambda_t^n(U) = \mathcal{R}_t$.

Demostración

According to Corollary 1, there exist a point $x \in U$ and a natural number $n_1 \in \mathbb{N}$ such that $\Lambda_t^{n_1}(x) = P_t$. Let $\mathcal{N}$ be a neighborhood of x such that $\mathcal{N} \subset U$, . Then there exists a set $\Theta_{-k} \subset \Lambda_t^{n_1}(\mathcal{N})$ and hence, $\Lambda_t^n(\mathcal{N}) = \mathcal{R}_t$ for some n.

Part VI

Some final comments

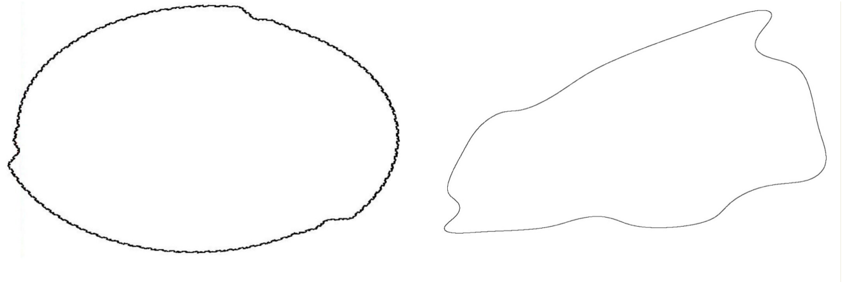
Last remarks

- In the paper submitted to Nonlinearity the Theorem was proved for the interval $\mathcal{I} = [0.88, 1]$. Recently the proof was extended to the interval $[1/\sqrt[3]{2}, 1]$. That is, for the case in which the attractor is connected (*Country Bread Attractor*).
- For the case when the attractor is not connected (*fairy cakes*) we have introduced a renormalization process. However, renormaliation forces to consider two-parametric family $\Lambda_{t,s}$ of EBM with matrix

$$A_{t,s} = \begin{pmatrix} t & s \\ t & -t \end{pmatrix}.$$

- The family $\Lambda_{t,s}$ furnish new types of attractors which were observed numerically for $T_{a(s),b(s)}$, but do not appear for the family Λ_t.

An example



On the left, the attractor for a given $\Lambda_{t,s}$. On the right, the attractor for $T_{a(s),b(s)}$ with $s = 1.8862$.

Outstanding work

1. To prove the transitivity of Λ_t for the remaining values of the parameter.
2. We know that for each $t \in \mathcal{I}$ there exists a unique absolutely continuous and ergodic invariant measure μ_t which is determined by a function f_t. But, is the map $t \hookrightarrow f_t$ continuous?
3. To bring the results obtained for the family of *EBMs* to the limit family $T_{a,b}$.
4. (For the afterlife)By using the possible results obtained for the limit family to prove the existence of two-dimensional strange attractor for the return map in a neighborhood of a generalized homoclinic tangency.

This work has been partially supported by MEC grant MTM2011-22956, MTM2012-32541 and CIRIT grant 2014SGR1145 and the Foundation for the promotion in Asturias of the scientific and technologic research BP1212-3.

Thanks very much!