Aubry Duality and a Thouless formula for quasi-periodic Schrödinger difference equations

Àlex Haro[1] & Joaquim Puig[2]

Grup de Sistemes Dinàmics UB-UPC
(1) Departament de Matemàtica Aplicada i Anàlisi (UB).
(2) Departament de Matemàtica Aplicada I (UPC).

New Perspectives in Discrete Dynamical Systems
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Quasi-periodic Schrödinger operators

In many physically relevant situations (eg. *quasi-crystals*, *Quantum Hall Effect*, *linearization around q-p orbits in DS*) it is necessary to consider a Schrödinger operator with quasi-periodic potential:

$$\left(H_{b,\omega,\phi}x\right)_n = x_{n+1} + x_{n-1} + bw_nx_n.$$

where $w_n = (W(\omega n + \phi))_{n\in\mathbb{Z}}$ is a quasi-periodic sequence with

- $W : \mathbb{T} = \mathbb{R}/\mathbb{Z} \to \mathbb{R}$ a real analytic funcion, called the potential.
- $\omega \in \mathbb{R}$ a irrational frequency,
- $\phi \in \mathbb{T}$ a phase.
- b is a coupling parameter.

and any of these operators satisfies

- It is bounded and self-adjoint on $l^2(\mathbb{Z})$.
- The spectrum is independent of ϕ since ω is irrational.

The dynamical connection: the eigenvalue equation

A link between spectral theory and dynamical systems for this equation is to consider the corresponding eigenvalue equation:

$$x_{n+1} + x_{n-1} + bW(n\omega + \phi)) x_n = a x_n, \qquad n \in \mathbb{Z}$$

where a is the energy. These are discrete difference equations and quasi-periodic versions of the classical Hill's equation

$$x''(t) + (a + bq(t)) x(t) = 0, \quad q(t) = q(t+T).$$

A “naive” discrete analog is the Harper equation

$$x_{n+1} + x_{n-1} + b\cos 2\pi (n\omega + \phi) x_n = a x_n, \qquad n \in \mathbb{Z},$$

the eigenvalue equation of the Almost Mathieu operator

$$\left(H_{b,\omega,\phi} x\right)_n = x_{n+1} + x_{n-1} + b\cos 2\pi (n\omega + \phi) x_n.$$

A dynamical perspective

These ev equations can be viewed as linear skew-products:

$$\underbrace{\begin{pmatrix} x_{n+1} \\ x_n \end{pmatrix}}_{v_{n+1}} = \underbrace{\begin{pmatrix} a - bW(\theta_n) & -1 \\ 1 & 0 \end{pmatrix}}_{A_{a,b}(\theta_n)} \underbrace{\begin{pmatrix} x_n \\ x_{n-1} \end{pmatrix}}_{v_n},$$

$$\theta_{n+1} = \theta_n + \omega \pmod 1,$$

and the solution is given by a cocycle on $SL(2,\mathbb{R}) \times \mathbb{T}$

$$M^{(N)}_{a,b,\omega}(\theta_0) = \begin{cases} M_{a,b}(\theta_{N-1}) \dots M_{a,b}(\theta_0) & N > 0, \\ I & N = 0, \\ M^{-1}_{a,b}(\theta_N) \dots M^{-1}_{a,b}(\theta_{-1}) & N < 0. \end{cases}$$

$$\left(A_{a,b}, 2\pi\omega\right)^n (I, \theta_0) = \left(M^{(N)}_{a,b,\omega}(\theta_0), \theta_0 + n\omega\right)$$

$$\begin{array}{rccc} (A_{a,b}, \omega) : & SL(2,\mathbb{R}) \times \mathbb{T} & \longrightarrow & SL(2,\mathbb{R}) \times \mathbb{T} \\ & (X, \theta) & \mapsto & \left(A_{a,b}(\theta) X, \theta + \omega\right). \end{array}$$

Another perspective (not new, though...)

Writing $y_n = x_{n-1}/x_n$, a family of Harper-like maps on $\bar{\mathbb{R}} \times \mathbb{T}$

$$\begin{cases} y_{n+1} & = \underbrace{\dfrac{1}{a - bW(\theta_n) - y_n}}_{f_{a,b}(y_n,\theta_n)}, \\ \theta_{n+1} & = \theta_n + \omega \qquad \pmod 1, \end{cases}$$

- $y \in \overline{\mathbb{R}} = [-\infty, +\infty]$ and $\theta \in \mathbb{T} = \mathbb{R}/\mathbb{Z}$.
- b (coupling), a (energy) and ω (irrational frequency) are parameters. Harper happens when $W = \cos$.

A Harper map is a skew-product map on $\overline{\mathbb{R}} \times \mathbb{T}$

$$F_{a,b,\omega}(y_n, \theta_n) = (f_{a,b}(y_n, \theta_n), \theta_n + \omega),$$

$$F^{(N)}_{a,b,\omega}(y_0, \theta_0) = (f^{(N)}_{a,b}(y_0, \theta_0), \theta_0 + N\omega), \quad N \in \mathbb{Z}.$$

. . . which is an qpf circle map

To get rid of the point at ∞, take polar coordinates $\varphi = \arctan y$ so that $y \in \mathbb{P} \simeq [-\pi/2, \pi/2]$ and the resulting equations are

$$\begin{cases} \varphi_{n+1} = \underbrace{\arctan\left(\dfrac{1}{a - b\cos(2\pi\theta_n) - \tan\varphi_n}\right)}_{\tilde{f}_{a,b}(\varphi_n,\theta_n)}, \\ \theta_{n+1} = \theta_n + \omega \pmod 1. \end{cases}$$

Since $\mathbb{P} \times \mathbb{T} \simeq \mathbb{S}^1$, it is a quasi-periodically forced circle map.

Take home message of the talk

Quasi-periodic Schrödinger operators (and their difference equations, skew-producs and maps)

- Display interesting and nontrivial phenomena: coexistence of different spectral types, nonuniform hyperbolicity or SNAs . . .
- Their study requires the combination of different areas.

An excursion in Etymology

SNA

Strange Nonchaotic Attractor

ANCE
Atractor No Caòtic Estrany

ANCE
Atractor No Caótico Extraño

An excursion in Etymology

DANCE

Dance Net

DINÁMICA, ATRACTORES Y NOLINEALIDAD
CAOS Y ESTABILIDAD

The DANCE Network: History

≈2000	Ll. Alsedà and A. Delshams became interested in SNA
2001	XT2001: Application for a Catalan network "Dinàmica discreta en dimensió baixa i atractors estranys" (Discrete Dynamics in low dimension and strange attractors), formed by 5 nodes (UAB, UB, UGR, UOV and UPC), unsuccessful.
2002	BFM2001: Spanish network DANCE (Dinámica no lineal en dimensión baja y atractores extraños), formed by 7 nodes (UAB, UB, UGR, UOV, UPC, US and UVA), successful!!!
2003	Ddays 2003 in Salou: Total discussion about SNA
2003	BFM2002: Spanish network DANCE (Dinámica, Atractores y Nolinealidad: Caos y Estabilidad), formed by 10 nodes (UAB, UB, UGR, UIB, UM, UOV, UPC, US, UV and UVA)
2004	RTNS 2004 in Palma de Mallorca

The DANCE network today

Today The DANCE network (http://www.dance-net.org/) has 21 nodes, more than 200 researchers

- Ddays 2003, 2004, 2006, 2008, 2010, 2012, 2014
- RTNS 2004, 2005, 2006, 2007, 2008, 2009, 2010, 2011, 2012, 2013, 2014
- There have been several other coordinators, but Lluís Alsedà is currently again one of the two coordinators
- The SNA topic has been studied by several people of the group: Lluís Alsedà, Sara Costa, Jordi-Lluís Figueras, Àlex Haro, Àngel Jorba, Carmen Núñez, Rafael Obaya, Joaquim Puig, Pau Rabassa, Joan Carles Tatjer, ...

DANCE rules

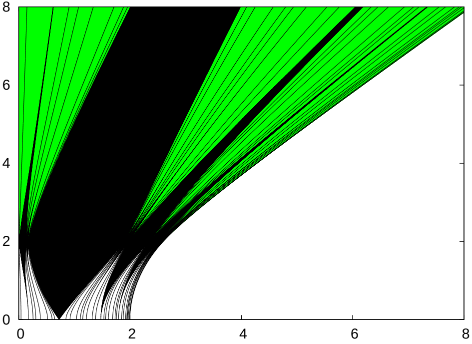
- Try new dances.
- Do not dance alone.

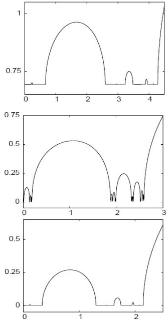
Lyapunov exponent. Almost Mathieu & beyond

(Upper) Lyapunov exponent of $x_{n+1} + x_{n-1} + bW(\theta_n)x_n = ax_n$

$$\gamma^H(a,b) = \lim_{N\to\infty} \frac{1}{N+1} \int_{\mathbb{T}} \log \left\| A_{a,b}(2\pi N\omega + \theta) \cdots A_{a,b}(\theta) \right\| d\theta$$

well-studied behaviour for the AMO $W(\theta) = \cos 2\pi\theta$. In the spectrum it equals $\max\left(0, \log \frac{|b|}{2}\right)$. For $b = 1, 2, 4$:





Towards an explanation for the AMO: the IDS

$$\kappa_{L,b,\omega,\phi}(a) = \frac{1}{L}\#\left\{\text{eigenvalues} \leq a \text{ of } H_{b,\omega,\phi}|_{\{1,\ldots,L\}}\right\}$$

for fixed with a, b and ϕ and some boundary conditions. Then

$$\lim_{L\to\infty} \kappa_{L,b,\omega,\phi}(a) = k_{b,\omega}(a),$$

is the integrated density of states (IDS), exists and satisfies

- it is independent of ϕ.
- it is continuous and not decreasing function of a (b fixed).
- $a \in \sigma(b,\omega)$, spectrum of $H_{b,\omega,\phi}$ $\Leftrightarrow$ the IDS increases at a.

One can recover the Lyapunov exponent from the IDS through the Thouless Formula, which holds for more general potentials:

$$\gamma^H(a,b) = \int_{\mathbb{R}} \log|a-a'|\, d\kappa_{b,\omega}(a') \text{ for } a \in \mathbb{C} \text{ and } b \in \mathbb{R}.$$

Why is the Almost Mathieu, $W = b\cos$, so special? I

The basic reason is invariance through Aubry Duality

- Assume that a is a point eigenvalue of an AMO $H_{b,\phi}$ whose e.f decays exponentially in $|n|$ (homoclinic at zero).
- This means that there is an exponentially decaying $(\psi_n)_n$, solution of the eigenvalue equation

$$\psi_{n+1} + \psi_{n-1} + b\cos 2\pi(\omega n + \phi)\psi_n = a\psi_n, \qquad n \in \mathbb{Z},$$

- Think of $(\psi_n)_n$ as the Fourier coefficients of an analytic funcion on $\mathbb{T}$ and consider the quasi-periodic Bloch wave

$$x_n = e^{\mathrm{i}n\phi}\tilde{\psi}(\omega n), \quad \tilde{\psi}(\theta) = \sum_{k\in\mathbb{Z}} \psi_k e^{\mathrm{i}k\theta}.$$

Why is the Almost Mathieu, $W = b\cos$, so special? II

- This sequence $(x_n)_{n\in\mathbb{Z}}$ satisfies the difference equation

$$\frac{b}{2}\left(x_{n+1} + x_{n-1}\right) + 2\cos(\omega n)x_n = ax_n,$$

an e.v. equation for the Almost Mathieu with parameters

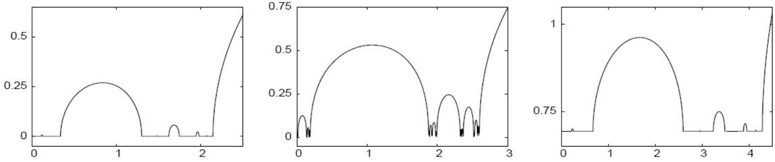
$$\beta = \frac{4}{b}, \quad \alpha = \frac{2a}{b}$$

- Aubry duality is this mechanism and applies to many situations. For example, the IDS is invariant through duality

$$\kappa(a,b) = \kappa\left(\frac{2a}{b}, \frac{4}{b}\right).$$

- Aubry duality can be made more precise under Diophantine conditions for ω and ϕ.

Duality of the Lyapunov exponent for the AMO



Using Thouless Formula for the Almost Mathieu

$$\gamma^H(a,b) = \int_{\mathbb{R}} \log|a - a'|d\kappa_{b,\omega}(a')$$

and the duality of the IDS, a change of variables shows that

$$\gamma^H(a,b) = \log\frac{|b|}{2} + \gamma^H\left(\frac{2a}{b}, \frac{4}{b}\right)$$

Since β^H is zero in the spectrum for $|b| \leq 2$, (nontrivial but natural, proved by Bourgain & Jitomirskaya 2002), in particular

$$\gamma^H(a,b) = \max\left(0, \log\frac{|b|}{2}\right), \quad a \in \sigma(b,\omega).$$

Extension to more general potentials I

- The previous trick works only for the Almost Mathieu.
- However, we can still use it for more general trigonometric polynomials.
- For example, consider a Schrödinger operator with a potential with two harmonics:

$$(hx)_n = x_{n+1} + x_{n-1} + 2\beta \underbrace{\left(\cos(2\pi\theta_n) + \cos(4\pi\theta_n)\right)}_{W(\theta_n)} x_n,$$

$$\theta_n = \theta_0 + \omega n, \quad n \in \mathbb{Z},$$

  with coupling $\beta \neq 0$.
- $W : \mathbb{T} \to \mathbb{R}$ could be any trigonometric polynomial.
- We can compute numerically the Lyapunov exponents.

Extension to more general potentials II

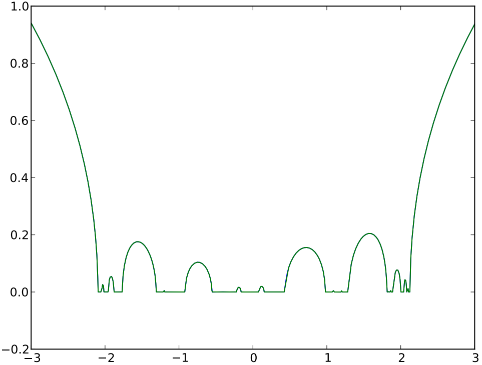


Figure : Upper Lyapunov exponent for $\beta = 0.25$ and $\omega = \frac{\sqrt{5}-1}{2}$.

Extension to more general potentials III

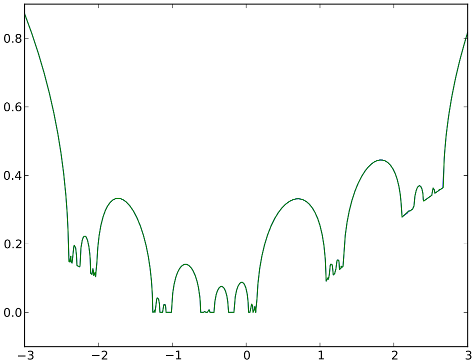


Figure : Upper Lyapunov exponent for $\beta = 0.5$ and $\omega = \frac{\sqrt{5}-1}{2}$.

Extension to more general potentials IV

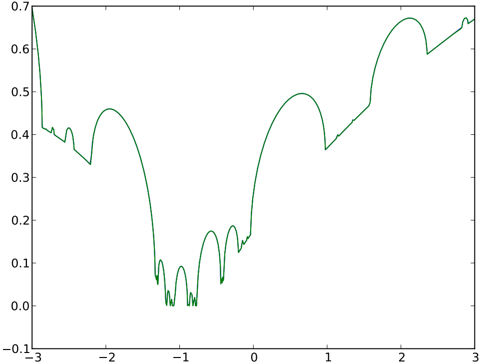


Figure : Upper Lyapunov exponent for $\beta = 0.75$ and $\omega = \frac{\sqrt{5}-1}{2}$.

Extension to more general potentials V

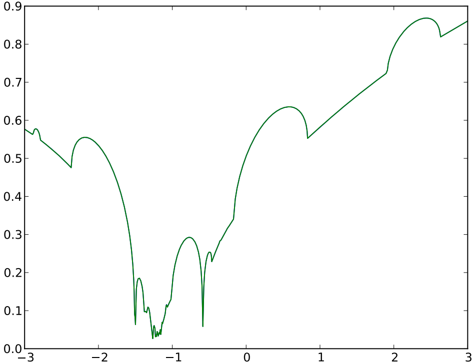


Figure : Upper Lyapunov exponent for $\beta = 1$ and $\omega = \frac{\sqrt{5}-1}{2}$.

More numerical explorations

J Dyn Diff Equat (2011) 23:649–669
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Resonance Tongues and Spectral Gaps in Quasi-Periodic Schrödinger Operators with One or More Frequencies. A Numerical Exploration

Joaquim Puig · Carles Simó

Questions

- For small values of β, the Lyapunov exponent is zero in the spectrum (well-understood by Eliasson's reducibility theory).
- For large values of β, the Lyapunov exponent is always positive ("easy" using Herman's trick).
- For intermediate values, coexistence (cf. Avila ICM2014).
- When the Lyapunov exponent is positive, it is not constant, but rather on a smooth curve (related to the *stratificated regularity* of the Lyapunov exponents (Avila 2013)).
- Is it possible to understand this through Aubry Duality?
- Can this explanation be purely dynamical?

An Aubry Duality approach I

- The Aubry dual of the operator with two harmonics

$$(hx)_n = x_{n+1} + x_{n-1} + 2\beta \underbrace{\left(\cos(2\pi\theta_n) + \cos(4\pi\theta_n)\right)}_{W(\theta_n)} x_n,$$

  is a difference operator of order 4 (twice the deg. of W)

$$(h'x)_n = \beta\left(x_{n+2} + x_{n+1} + x_{n-1} + x_{n-2}\right) + 2\cos(2\pi\theta_n)x_n, \; n \in \mathbb{Z}.$$

- For any α, its eigenvalue equation $h'x = \alpha x$, defines a skew-product on $\mathbb{R}^4$,

$$\begin{pmatrix} x_{n+2} \\ x_{n+1} \\ x_n \\ x_{n-1} \end{pmatrix} = \begin{pmatrix} -1 & \frac{\alpha}{\beta} - \frac{2}{\beta}\cos(2\pi\theta_n) & -1 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} x_{n+1} \\ x_n \\ x_{n-1} \\ x_{n-2} \end{pmatrix},$$

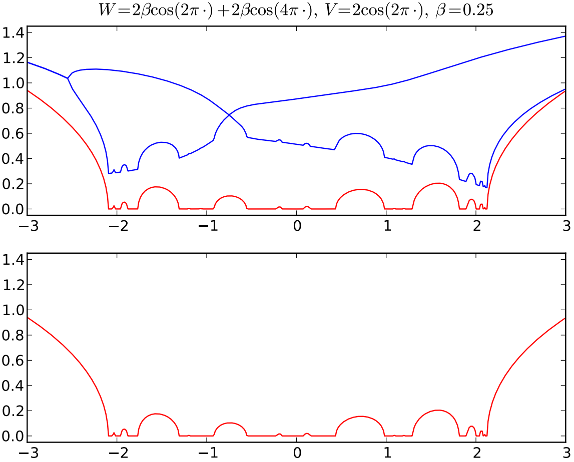
$$\theta_{n+1} = \theta_n + \omega.$$

An Aubry Duality approach II

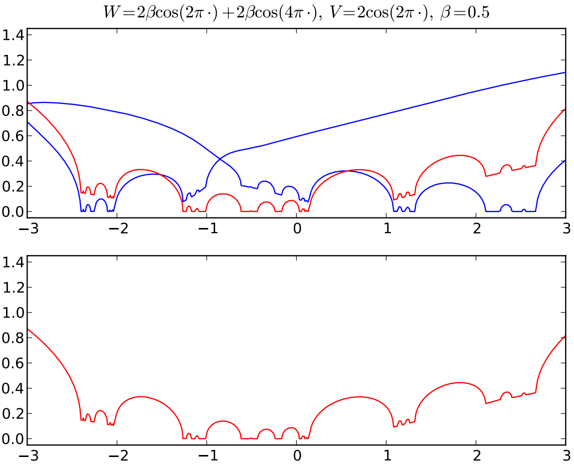
- Unlike the Almost Mathieu, these linear skew-products are not in $SL(2,\mathbb{R})$ but they preserve an adapted complex symplectic structure (dependent on W.)
- In particular, for any $\alpha \in \mathbb{C}$ and $\beta \neq 0$, it has 2 Lyapunov exponents which are non-negative, $\gamma_1(\alpha,\beta)$ and $\gamma_2(\alpha,\beta)$.
- Which is the relationship between the Lyapunov exponent of the original Schrödinger operator h at α and these two Lyapunov exponents of the dual?
- We can show a similar relation than for the AMO:

$$\gamma^{h}(\alpha,\beta) = \underbrace{\gamma_1^{h'}(\alpha,\beta) + \gamma_2^{h'}(\alpha,\beta) + \log|\beta|}_{\text{normalized Lyapunov semi-trace}}, \tag{1}$$

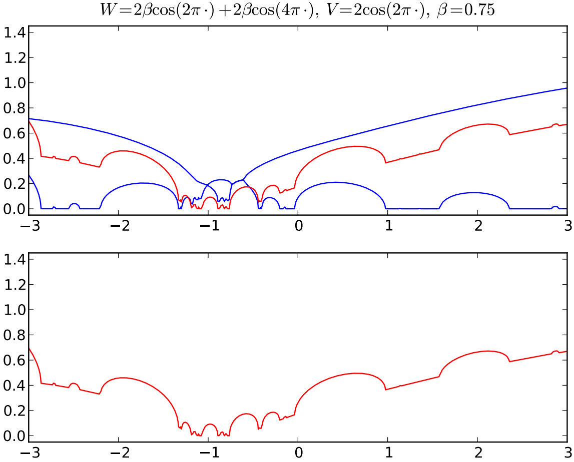
Numerical examples I



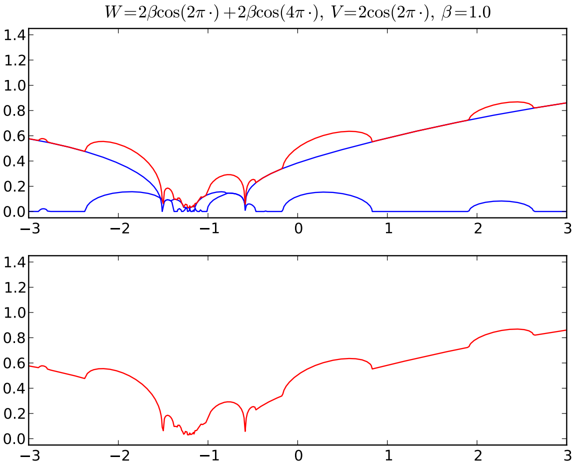
Numerical example



Numerical example



Numerical example



Quasi-periodic long-range operators

A quasi-periodic long-range operator of finite range d and (irrational) frequency $\omega \in \mathbb{R}$ is a bounded self-adjoint operator $h = h_{V,W,\theta_0}$ acting on $x = (x_n)_n \in \ell^2(\mathbb{Z}, \mathbb{C})$ by

$$(hx)_n = \sum_{k=-d}^{d} V_k x_{n+k} + W(\theta_n) x_n, \quad n \in \mathbb{Z},$$

where

- $V : \mathbb{T} \to \mathbb{R}$ is a real trigonometric function with average 0, the symbol, with Fourier representation

$$V(\theta) = \sum_{k=-d}^{d} V_k e^{2\pi i k\theta};$$

- $W : \mathbb{T} \to \mathbb{R}$ is a real analytic funcion, the potential;
- $\theta_0 \in \mathbb{T}$ is a phase, and $\theta_n = \theta_0 + n\omega$ for $n \in \mathbb{Z}$.

Quasi-periodic long-range linear skew-products

The eigenvalue equation of h for $\alpha \in \mathbb{C}$,

$$\sum_{k=-d}^{d} V_k x_{n+k} + W(\theta_n) x_n = \alpha x_n, \quad n \in \mathbb{Z},$$

is equivalent to

$$\underbrace{\begin{pmatrix} x_{n+d} \\ \vdots \\ x_{n+2} \\ x_{n+1} \\ x_n \\ \vdots \\ x_{n-d+2} \\ x_{n-d+1} \end{pmatrix}}_{u_{n+1}} = \underbrace{\frac{1}{V_d} \left(\begin{array}{cccc|cccc} -V_{d-1} & \dots & -V_1 & \alpha - W(\theta_n) & -V_{-1} & \dots & -V_{-d+1} & -V_{-d} \\ V_d & & & & & & & \\ & \ddots & & & & & & \\ & & V_d & & & & & \\ \hline & & & V_d & & & & \\ & & & & V_d & & & \\ & & & & & \ddots & & \\ & & & & & & V_d & \end{array} \right)}_{A^h_\alpha(\theta_n)} \underbrace{\begin{pmatrix} x_{n+d-1} \\ \vdots \\ x_{n+1} \\ x_n \\ x_{n-1} \\ \vdots \\ x_{n-d+1} \\ x_{n-d} \end{pmatrix}}_{u_n}$$

$$\theta_{n+1} = \theta_n + \omega,$$

defining a *long-range linear skew-product*
$(A^h_\alpha, \tau) : \mathbb{C}^{2d} \times \mathbb{T} \to \mathbb{C}^{2d} \times \mathbb{T}$.

Symplectic properties of long-range cocycles

An adapted complex symplectic structure

Proposition

For $\alpha \in \mathbb{R}$, the long-range skew-product (A_α^h, τ) is complex symplectic with respect to the complex symplectic structure

$$\Omega = \begin{pmatrix} 0 & -C^* \\ C & 0 \end{pmatrix},$$

where

$$C = \begin{pmatrix} V_d & \cdots & V_1 \\ 0 & \ddots & \vdots \\ 0 & 0 & V_d \end{pmatrix}.$$

That is:

$$1(A_\alpha^h(\theta))^* \, \Omega \, A_\alpha^h(\theta) = \Omega.$$

Compare with [Johnson 87].

Thouless formula for long-range operators

Generalization of previous objects

The integrated densitity of states of the operator h is a non-decreasing function $\kappa_h : \mathbb{R} \to [0, 1]$ defined as the limit

$$\kappa_h(a) = \lim_{N\to\infty} \frac{1}{2N+1} \# \left\{ \text{eigenvalues} \le a \text{ of } h^{[-N,N]} \right\},$$

where $h^{[-N,N]}$ is the restriction of h to the interval $[-N, N]$ with zero boundary conditions.

The normalized Lyapunov semi-trace or entropy of h at α, $\bar{\gamma}^h(\alpha)$ is

$$\bar{\gamma}^h(\alpha) = \gamma_1^h(\alpha) + \ldots + \gamma_d^h(\alpha) + \log(V_d),$$

where $\gamma_1^h(\alpha) \ge \ldots \ge \gamma_d^h(\alpha) \ge 0$ are the d non-negative Lyapunov exponents of (A_α^h, τ).

Thouless formula for long-range operators

The main result

Theorem (Thouless formula)

The following integral formula holds:

$$\int_{\mathbb{R}} \log|\alpha - a| d\kappa_h(a) = \bar{\gamma}^h(\alpha),$$

where κ_h is the IDS of the long-range operator h, and $\bar{\gamma}^h(\alpha)$ is the normalized Lyapunov semi-trace.