On completely scrambled systems

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(joint work with M. Foryś, W. Huang and J. Li)

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Basic notation

1. (X, d) - compact metric space
2. $f : X \to X$ - continuous (denoted $f \in C(X)$)
3. (X, f) - dynamical system

Scrambled sets

1. (x, y) is a LY-pair if it is proximal but not asymptotic, that is, respectively,

$$\begin{aligned}\liminf_{n\to\infty} d(f^n(x), f^n(y)) &= 0 \quad \text{and}\\ \limsup_{n\to\infty} d(f^n(x), f^n(y)) &> 0.\end{aligned}$$

2. δ-LY when $\limsup d(f^n(x), f^n(y)) > \delta$

3. S is a scrambled set (resp. δ-scrambled) if every pair od distinct points is LY (resp. δ-LY).

Some history

1. Start of the story...
   - [1975, Li & Yorke] If $f\colon [0,1] \to [0,1]$ has a point of period 3 then it has a scrambled set.

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   - [1975, Li & Yorke] If $f\colon [0,1] \to [0,1]$ has a point of period 3 then it has a scrambled set.
2. And other classical results implying existence of scrambled set...
   - [Iwanik, 1989] Weak mixing
   - [Huang & Ye, 2002] Devaney chaos
   - [Blanchard, Glasner, Kolyada & Maas, 2002] Positive entropy

Natural questions

Question

What is the size of scrambled sets in concrete cases/spaces?

Question

How large scrambled sets can be?

How big scrambled set can be in dimension n?

1. Traditionally scrambled set should at least be uncountable (to indicate complicated dynamics).
2. By result of Misiurewicz, on $[0,1]$ scrambled set can have full (Lebesgue) measure.
3. The same is true on $[0,1]^n$ (first proved by Kato).

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4. But $\mathcal{C}^1$-maps on the unit interval never have full measure scrambled sets (Jimenez-Lopez).
5. And on interval (continuous case) scrambled set is never residual (Bruckner & Hu; Gedeon). The same holds on topological graphs (Mai).

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5. And on interval (continuous case) scrambled set is never residual (Bruckner & Hu; Gedeon). The same holds on topological graphs (Mai).
6. Mai constructed a DS on $(0,1)^n$ with all non-diagonal pairs LY and conjectured that it is not possible on compact spaces (the case $[0,1]^n$, $n>1$ is probably still open (?)).

Definition

A dynamical system (X,f) is completely scrambled if X is a scrambled set.

Completely scrambled systems

1. If (X, f) is transitive and completely scrambled then it is a homeomorphism.
2. (e.g. Akin & Kolyada) The following conditions are equivalent:
    1. (X, f) is proximal, i.e. $\liminf_n d(f^n(x), f^n(y)) = 0$ for every $x, y \in X$,
    2. (X, f) has a fixed point which is the unique minimal subset of X.

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   2. (X, f) has a fixed point which is the unique minimal subset of X.
3. (e.g. King) If X is infinite and f is homeomorphism, then for every $\varepsilon > 0$
   1. there are points $x \neq y$ with ε-bounded future, i.e.
   2. $d(f^n(x), f^n(y)) < \varepsilon$ for every $n \geq 0$.
4. In particular, for any $\delta > 0$ the space X cannot be δ-scrambled (there is no completely δ-scrambled system)
5. and expansive maps are never completely scrambled (so no example among shift spaces on finite alphabet).

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5. and expansive maps are never completely scrambled (so no example among shift spaces on finite alphabet).
6. δ-scrambled set S can be closed or invariant (i.e. $f(S) \subset S$).

Method of Huang and Ye (publ. 2001; results from 1999)

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Homeomorphisms with the whole compacta being scrambled sets

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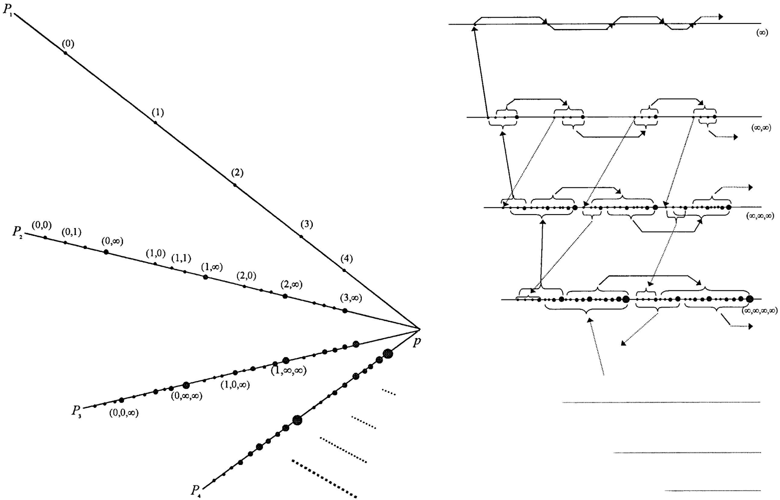
1. There is a countable and compact set $Y \subset \mathbb{R}^2$ admitting completely scrambled (homeomorphism) F.
2. Technique for extending dimension....
   - Let (Y, F) be completely scrambled and let p be the unique fixed point of F in Y.
   - Take any compact set Z and let g be obtained form the map $F \times \mathrm{id}_Z$ by collapsing to a point $(Y \times \{z_0\}) \cup (\{p\} \times Z)$ in $Y \times Z$ for some $z_0 \in Z$.
   - If Z is continuum then resulting space is also a continuum.
3. DS obtained by this method is never transitive.

Brief overview on the construction



Brief overview on the construction

$$(n_1) \longrightarrow (n_1+1), \tag{P$_0$}$$

$$(n_1, n_2) \longrightarrow (n_1+1, n_2-1), \quad n_2 \geq 1, \tag{P$_1$}$$

$$(0,0) \longrightarrow (0), \tag{P$_2$}$$

$$(n_1, n_2, n_3) \longrightarrow (n_1+1, n_2-1, n_3), \quad n_2 \geq 2, \tag{P$_3$}$$

$$(n_1, 1, n_3) \longrightarrow (n_1+1, 0, n_3+1), \tag{P$_4$}$$

$$(0, 0, n_3) \longrightarrow (0, n_3), \tag{P$_5$}$$

$$(n_1+1, 0) \longrightarrow (n_1, 0, 0), \tag{P$_6$}$$

$$(n_1, n_2, n_3, \dots, n_k) \longrightarrow (n_1+1, n_2-1, n_3, \dots, n_k), \quad n_2 \geq 2, k \geq 4, \tag{P$_7$}$$

$$(n_1, 1, n_3, n_4, \dots, n_k) \longrightarrow (n_1+1, 0, n_3+1, n_4, \dots, n_k), \quad k \geq 4, \tag{P$_8$}$$

$$(0, 0, n_3, n_4, \dots, n_k) \longrightarrow (0, n_3, n_4, \dots, n_k), \quad n_3 \geq 1, k \geq 4, \tag{P$_9$}$$

$$(0, 0, 0, n_4, \dots, n_k) \longrightarrow (0, 0, n_4+1, \dots, n_k), \quad k \geq 4, \tag{P$_{10}$}$$

$$(n_1+1, 0, n_4, \dots, n_k) \longrightarrow (n_1, 0, 0, n_4, \dots, n_k), \quad k \geq 4. \tag{P$_{11}$}$$

$$g(n_1, \dots, n_k, \overbrace{\infty, \dots, \infty}^{j}) = (m_1, \dots, m_l, \overbrace{\infty, \dots, \infty}^{j})$$

Hierarchy of (topological) "mixing" properties

1. f is transitive if for every nonempty open set U, V there is n such that $f^n(U) \cap V \neq \emptyset$.
2. f is totally transitive if f^n is transitive for every n.
3. f is (topologically) weakly mixing if $f \times f$ is transitive. Equivalently: $f^n(U_i) \cap V_i \neq \emptyset$ for $i = 1, \ldots, m$, $n, m > 0$.
4. f is (topologically) mixing if there is N such that $f^n(U) \cap V \neq \emptyset$ for $n > N$.

$$\text{Mix} \implies \text{WMix} \implies \text{Tot. Trans} \implies \text{Trans}$$

Questions in H&Y paper

The following questions are form H & Y paper (and probably from 1999).

1. Can completely scrambled system have positive topological entropy?

2. Can completely scrambled system be transitive?

Questions in H&Y paper

The following questions are form H & Y paper (and probably from 1999).

1. Can completely scrambled system have positive topological entropy?
   - NO!
   - Every system with positive entropy has an asymptotic pair (Blanchard, Host, Ruette, 2002)
2. Can completely scrambled system be transitive?
   - YES! (Katznelson & Weiss construction, 1981; Akin, Auslander & Berg, 1996)
3. Both answers added to H & Y paper before final publication.

Conjecture

Completely scrambled examples with mixing properties should exist.

Related question

Is there transitive completely scrambled system (on continua) in every dimension?

Uniformly rigid systems

1. Dynamical system (X, f) is uniformly rigid if for every $\varepsilon > 0$ there is $n > 0$ such that $d(x, f^n(x)) < \varepsilon$ for every $x \in X$.
2. Natural examples:
   - Periodic orbit
   - Irrational rotation of the circle (or $\mathbb{T}^n$).
3. Uniformly rigid systems cannot be mixing.
4. Uniformly rigid systems do not contain (proper) asymptotic pairs.

Uniformly rigid systems

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4. Uniformly rigid systems do not contain (proper) asymptotic pairs.
5. But uniformly rigid system can be weakly mixing (Glasner & Maon, 1989)
   - On various spaces of the form $\mathbb{S}^1 \times Y$ (including all $\mathbb{T}^n$, $n \geq 2$) there exists weakly mixing, uniformly rigid, minimal dynamical system.

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Theorem (Glasner & Maon)

Let $\mathcal{H}_0(Y)$ be a path component of identity in $\mathcal{H}(Y)$. If Y is nontrivial and action of $\mathcal{H}_0(Y)$ is minimal on Y then there is weakly mixing, uniformly rigid and minimal homeomorphism (on $\mathbb{S}^1 \times Y$) in

$$\overline{\{G^{-1} \circ (R_\alpha \times \mathrm{id}) \circ G : G \in \mathcal{H}(\mathbb{S}^1 \times Y)\}}$$

Uniformly rigid systems (Katznelson & Weiss example)

1. If (X, f) is uniformly rigid and proximal then it is completely scrambled.
2. Katznelson and Weiss provided a method of construction of uniformly rigid, proximal, transitive systems.
3. Constructed system is a subset of the Hilbert cube $[0,1]^{\mathbb{N}}$ with metric

$$d(\alpha, \beta) = \sum_{n=0}^{\infty} \frac{|\alpha(n) - \beta(n)|}{2^n}$$

and left shift $\sigma(\alpha)(i) = \alpha(i+1)$ on it.

Uniformly rigid systems (Katznelson & Waiss example)

1. Fix $L \geq 2$ and start with function $a_0\colon [-1,1] \to [0,1]$ such that

$$|a_0(s_1) - a_0(s_2)| \leq L|s_1 - s_2| \quad \text{and } a_0(1) = a_0(-1) = 1, \quad a_0(0) \neq 1.$$

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2. Define $a_1 \colon \mathbb{R} \to [0, 1]$ by $a_1(s) = a_0(s)$ when $|s| \leq 1$, and $a_1(s + 2) = a_1(s)$ for all $s \in \mathbb{R}$.
3. Put $a_p(s) = a_1(s/p)$, i.e. "stretch" graph of a_1.
4. Finally $a_\infty(s) = \sup_n a_{p_n}(s)$ for a sequence p_n where $p_{n+1} = p_n k_n$, $\{k_n\}_{n=1}^{\infty}$ is strictly increasing and 8 divides each k_n.

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5. Define $\alpha(n) = a_\infty(n)$ and $\mathbb{X} = \overline{\{\sigma^n(\alpha) : n \geq 0\}} \subset [0,1]^{\mathbb{N}}$.

Uniformly rigid systems (Katznelson & Waiss example)

Main features of K-W construction:

1. $|a_p(s_1) - a_p(s_2)| \leq |a_1(s_1/p) - a_1(s_2/p)| \leq \frac{L}{p}|s_1 - s_2|$ hence

$$|a_\infty(s + 2p_i) - a_\infty(s)| \leq \sup_{j>i} |a_j(s + 2p_m) - a_j(s)| \leq \sup_{j>i} \frac{2Lp_i}{p_j} \leq \frac{2L}{k_i}.$$

2. for any $\varepsilon > 0$, all odd m and $|s| < \varepsilon p_i / L$ we have

$$1 - \varepsilon < a_\infty(mp_i + s) \leq 1.$$

Theorem

Dynamical system $(\mathbb{X}, \sigma)$ (which is orbit closure of α) is uniformly rigid, and $\{\theta\}$ is its unique minimal subsystem, where $\theta(n) = 1$ for all n.

Theorem (Akin, Auslander, Berg)

If a_0 has strict minimum in 0 then $(\mathbb{X}, \sigma)$ is almost equicontinuous.

Uniformly rigid systems (Katznelson & Waiss example)

Theorem (Akin, Auslander, Berg)

Every almost equicontinuous dynamical system is uniformly rigid.

1. Defining $a_0(t) = 0$ for $|t| \leq 1/2$ and $a_0(t) = 2|t| - 1$ for $|t| > 1/2$ we obtain $(\mathbb{X}, \sigma)$ which is not almost equicontinuous.

Theorem

Let (X, f) be transitive and pointwise recurrent. If (X, f) contains a minimial set that is connected, then X is connected.

1. Then $\mathbb{X}$ is always connected. However dimension of $\mathbb{X}$ is unknown.

Weak mixing completely scrambled system - method I

This is part of joint work with M. Foryś, W. Huang and J. Li.

1. (X, f) is scattering if $(X \times Y, f \times g)$ is transitive for every minimal (Y, g).

Theorem (Akin & Glasner)

There exists scattering almost equicontinuous transitive DS (Y, g).

1. Combining various results from Akin-Glasner it can be proved that:
   - there exists extension $\pi\colon (Z, h) \to (Y, g)$,
   - (Z, h) is weakly mixing and uniformly rigid,
   - all minimal systems in (Z, h) are singletons.
2. Hence it is enough to collapse fixed points of (Z, h) to a single point.

Weak mixing completely scrambled system - method II

Lemma

Assume that (X, f) is transitive and that $x \in X$ has dense orbit. Then (X, f) is weakly mixing if and only if for any open neighborhood U of x there is $n > 0$ such that $n, n+1 \in N(U, U)$.

$$a_0(t) = \begin{cases} 0, & \text{for } t \in [-1, 1], \\ 1, & \text{for } t \in [-3, -2] \cup [2, 3], \\ -t-1, & \text{for } t \in [-2, -1], \\ t-1, & \text{for } t \in [1, 2] \\ 0, & \text{for } t \notin [-3, 3]. \end{cases}$$

- $p_0 = 3$, $L_n \geq p_{n-1}^2$, $p_n = p_0^2 L_n p_{n-1}$.
- $b_n(t) = a_n(t)$ for $t \in [-p_n, p_n]$, and $b_n(t + 2p_n) = b_n(t)$

Weak mixing completely scrambled system - method II

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- $b_n(t) = a_n(t)$ for $t \in [-p_n, p_n]$, and $b_n(t + 2p_n) = b_n(t)$
- $c_n(t) = b_0(t/(p_{n-1}L_n))$

$$a_{n+1}(t) = \begin{cases} \max\{b_n(t), c_{n+1}(t)\}, & \text{for } t \in [-p_{n+1}, p_0 L_{n+1} p_n], \\ \max\{b_n(t+1), c_{n+1}(t)\}, & \text{for } t \in (p_0 L_{n+1} p_n, p_{n+1}], \\ 0, & \text{for } t \notin [-p_{n+1}, p_{n+1}]. \end{cases}$$

- $a_\infty(t) = \sup_{n \in \mathbb{N}_0} a_n(t), \qquad$ for $t \in \mathbb{R}$.

Questions (unanswered so far)

1. Is it possible to construct transitive completely scrambled system in every dimension?
2. Examples of Huang & Ye are not pairwise recurrent. Are there pairwise recurrent and proximal systems in every dimension?
3. Can completely scrambled system be mixing?
4. Is it possible to characterize continua without completely scrambled homeomorphisms?
    - On dendrites there is no completely scrambled homeomorphism (Naghmouchi)

A related problem

1. We know that there is not completely δ-scrambled system.
2. When does invariant (not closed) δ-scrambled set exists?
3. We proved (with Balibrea and Garcia Guirao) that:
   - if (X, f) is mixing then there exists a dense Mycielski (i.e. countable sum of Cantor sets) invariant δ-scrambled set S (i.e. $f(S) \subset S$)
   - if (X, f) is weakly mixing then there exists a dense Mycielski invariant scrambled set S.
4. Is it only technical difficulty?

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   - if (X, f) is weakly mixing then there exists a dense Mycielski invariant scrambled set S.
4. Is it only technical difficulty?

Theorem (Foryś, Huang, Li & O.)

Let (X, f) be a non-trivial transitive dynamical system with a fixed point. The following conditions are equivalent:

1. (X, f) has a dense Mycielski invariant δ-scrambled set for some $\delta > 0$,
2. (X, f) has a fixed point and is not uniformly rigid.