Hyperbolicity and Compactness

Anosov Diffeomorphisms in the Plane

Zbigniew Nitecki[1]
(joint with Jorge Groisman[2])

[1]Tufts University
Medford, MA, USA

[2]IMERL
Montevideo, Uruguay

NPDDS 2014, Tossa de Mar
October 3, 2014

Outline

Outline

Outline

Outline

Outline

For Lluis

Feliz Cumpleaños, Amigo

For Lluis

Feliz Cumpleaños, Amigo (Viejo)!


The Meaning of Life

The Meaning of Life

La vida es una milonga
(Anibal Troilo,
Pa' Que Bailen Los Muchachos)

The Meaning of Life

La vida es una milonga
(Anibal Troilo,
Pa' Que Bailen Los Muchachos)
...y la dinámica hiperbólica es el Nuevo Tango

Compact Dynamics

Classically (i.e., since the 1960's) dynamical systems live in a compact phase space (e.g.,closed manifold).

Compact Dynamics

Classically (i.e., since the 1960's) dynamical systems live in a compact phase space (e.g.,closed manifold).
This automatically insures certain features of the dynamics:

Compact Dynamics

Classically (i.e., since the 1960's) dynamical systems live in a compact phase space (e.g.,closed manifold).
This automatically insures certain features of the dynamics:

- All orbits have nonempty α- and ω-limit sets.

Compact Dynamics

Classically (i.e., since the 1960's) dynamical systems live in a compact phase space (e.g.,closed manifold).
This automatically insures certain features of the dynamics:

- All orbits have nonempty α- and ω-limit sets.
- The notion of an attractor can be formulated in purely topological terms.

Compact Dynamics

Classically (i.e., since the 1960's) dynamical systems live in a compact phase space (e.g.,closed manifold).
This automatically insures certain features of the dynamics:

- All orbits have nonempty α- and ω-limit sets.
- The notion of an attractor can be formulated in purely topological terms.
- There is a unique uniform structure, so that many definitions in terms of a (Riemann) metric are independent of the metric used, and many can be formulated in purely topological terms.

Non-Compact Dynamics

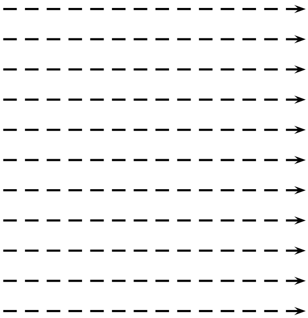
When the phase space is not compact, none of these are guaranteed.

Non-Compact Dynamics: Limit points

Orbits may have no α- or ω-limits points

Non-Compact Dynamics: Limit points

Orbits may have no α- or ω-limits points –in fact there may be **no** ω-limit points, for example in a parallel translation:



Attractors

If one tries to define the notion of an attractor in purely topological terms

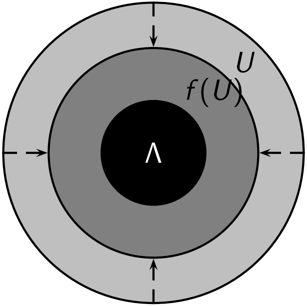
Attractors

If one tries to define the notion of an attractor in purely topological terms –for example:

Attractors

There exists a neighborhood U of the given (closed) set Λ such that

- $\operatorname{clos} f(U) \subset U$
- $\bigcap_{k=0}^{\infty} f^k(U)$

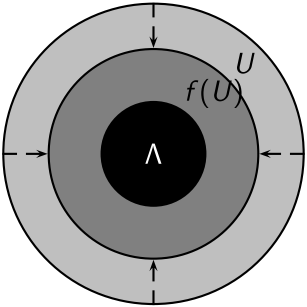


Attractors

There exists a neighborhood U of the given (closed) set Λ such that

- $\operatorname{clos} f(U) \subset U$
- $\bigcap_{k=0}^{\infty} f^k(U)$

The definition works well for compact attractors,

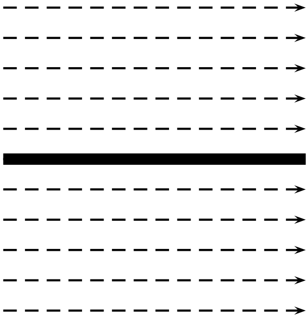


Attractors

There exists a neighborhood U of the given (closed) set Λ such that

- $\operatorname{clos} f(U) \subset U$
- $\bigcap_{k=0}^{\infty} f^k(U)$

The definition works well for compact attractors, but in the non-compact setting, for example, one finds that every closed set invariant under a parallel translation

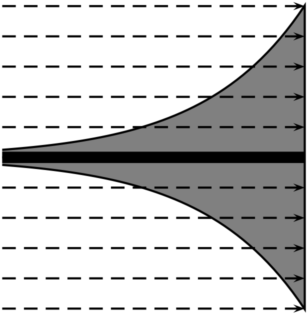


Attractors

There exists a neighborhood U of the given (closed) set Λ such that

- $\operatorname{clos} f(U) \subset U$
- $\bigcap_{k=0}^{\infty} f^k(U)$

The definition works well for compact attractors, but in the non-compact setting, for example, one finds that every closed set invariant under a parallel translation is **an attractor**

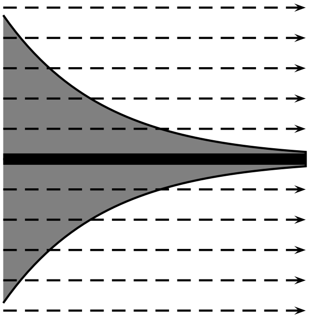


Attractors

There exists a neighborhood U of the given (closed) set Λ such that

- $\operatorname{clos} f(U) \subset U$
- $\bigcap_{k=0}^{\infty} f^k(U)$

The definition works well for compact attractors, but in the non-compact setting, for example, one finds that every closed set invariant under a parallel translation is **an attractor as well as a repeller (=attractor for** f^{-1}**)**.



Uniform Structures

Perhaps the most interesting difference between compact and non-compact systems is the important role played by the choice of a **uniform structure**.

Uniform Structures

Perhaps the most interesting difference between compact and non-compact systems is the important role played by the choice of a **uniform structure**. This is the particular aspect of a metric which allows us talk about two orbits being **asymptotic** when neither has an ω-limit point.

Anosov Diffeomorphisms (compact setting)

Recall that a diffeomorphism $f : M \to M$ of a compact manifold to itself is an **Anosov diffeomorphism** if it has

Anosov Diffeomorphisms (compact setting)

Recall that a diffeomorphism $f : M \to M$ of a compact manifold to itself is an **Anosov diffeomorphism** if it has a (global)

Hyperbolic Splitting

For every $x \in M$, the tangent space splits as

$$T_x M = E^s_x \oplus E^u_x$$

satisfying, for some constants $\mu < 1 < \lambda$ and some Riemannian metric $\| \cdot \|$,

- the splitting is f-invariant: for $\sigma = u$ and $s, Tf(E^\sigma_x) = E^\sigma_{f(x)}$;
- for every $\vec{v}_s \in E^s_x$, $\|Tf_x(\vec{v}_s)\| \leq \mu \|\vec{v}_s\|$;
- for every $\vec{v}_u \in E^u_x$, $\|Tf_x(\vec{v}_u)\| \geq \lambda \|\vec{v}_u\|$.

Anosov Dynamics

In a compact setting, the existence of a global hyperbolic splitting has major dynamic consequences, including

Anosov Dynamics

In a compact setting, the existence of a global hyperbolic splitting has major dynamic consequences, including

- density of periodic orbits

Anosov Dynamics

In a compact setting, the existence of a global hyperbolic splitting has major dynamic consequences, including

- density of periodic orbits
- existence of a dense orbit

Anosov Dynamics

In a compact setting, the existence of a global hyperbolic splitting has major dynamic consequences, including

- density of periodic orbits
- existence of a dense orbit
- sensitive dependence on initial conditions.

Anosov Dynamics

In a compact setting, the existence of a global hyperbolic splitting has major dynamic consequences, including

- density of periodic orbits
- existence of a dense orbit
- sensitive dependence on initial conditions.

This is often taken as a definition of "chaos".

Stable and Unstable Foliations

For us, the most important property of Anosov diffeomorphisms is that the two bundles E^s and E^u are automatically integrable: there is a **stable foliation** (respectively **unstable foliation**) whose leaves are tangent at each of its points to E^s (respectively E^u).

Stable and Unstable Foliations

For us, the most important property of Anosov diffeomorphisms is that the two bundles E^s and E^u are automatically integrable: there is a **stable foliation** (respectively **unstable foliation**) whose leaves are tangent at each of its points to E^s (respectively E^u).

Furthermore, the stable leaf through any point x is its **stable manifold**, defined dynamically as the set of points y such that $dist(f^n(x)), f^n(y)) \to 0$.

Hyperbolic Splitting

When the phase space is non-compact (our example will be $\mathbb{R}^2$), we adopt the definition of hyperbolic splitting that we had in the compact case, with the proviso that the metric be complete:

Hyperbolic Splitting

When the phase space is non-compact (our example will be $\mathbb{R}^2$), we adopt the definition of hyperbolic splitting that we had in the compact case, with the proviso that the metric be complete:

Hyperbolic Splitting (Non-Compact Setting)

For every $x \in M$, the tangent space splits as

$$T_x M = E_x^s \oplus E_x^u$$

satisfying, for some constants $\mu < 1 < \lambda$ and some **complete** Riemannian metric $\|\cdot\|$,

- the splitting is f-invariant: for $\sigma = u$ and $s, Tf(E_x^\sigma) = E_{f(x)}^\sigma$;
- for every $\vec{v}_s \in E_x^s$, $\|Tf_x(\vec{v}_s)\| \leq \mu\|\vec{v}_s\|$;
- for every $\vec{v}_u \in E_x^u$, $\|Tf_x(\vec{v}_u)\| \geq \lambda\|\vec{v}_u\|$.

Anosov Structure

The **Stable Manifold Theorem**, which ensures the integrability of the stable and unstable bundles, rests on some uniformity estimates for derivatives which are not automatic in a non-compact setting. Wishing to avoid this issue, we build the foliations into our definition.

Anosov Structure

The **Stable Manifold Theorem**, which ensures the integrability of the stable and unstable bundles, rests on some uniformity estimates for derivatives which are not automatic in a non-compact setting. Wishing to avoid this issue, we build the foliations into our definition.

Anosov Structure

An **Anosov structure** for a diffeomorphism $f : \mathbb{R}^2 \to \mathbb{R}^2$ consists of a complete Riemannian metric $\|\cdot\|$ on $\mathbb{R}^2$, two constants $\mu < 1 < \lambda$, and a pair $\mathcal{F}^s$, $\mathcal{F}^u$ of transverse foliations of $\mathbb{R}^2$ by curves satisfying:

- f takes the leaves of each foliation to other leaves;
- If $\vec{v}_s$ is tangent to a leaf of $\mathcal{F}^s$, then $\|Tf(\vec{v}_s)\| \leq \mu\|\vec{v}_s\|$;
- If $\vec{v}_u$ is tangent to a leaf of $\mathcal{F}^u$, then $\|Tf(\vec{v}_u)\| \geq \lambda\|\vec{v}_u\|$.

Linear Hyperbolic Maps

An obvious example of a diffeomorphism of $\mathbb{R}^2$ with an Anosov structure is the action of any 2×2 matrix with eigenvalues $\mu < 1 < \lambda$: the standard Euclidean metric and the foliations by translates of the two eigenspaces give the Anosov structure. In particular, any two such diffeomorphisms are topologically conjugate, so we will take as our basic instance of this example the matrix

$$\begin{pmatrix} 2 & 0 \\ 0 & \frac{1}{2} \end{pmatrix}$$

for which $\mathcal{F}^s$ (resp. $\mathcal{F}^u$) is the foliation by vertical (resp. horizontal) lines.

Warren White's Example

In 1971, Warren White constructed another, strikingly counter-intuitive example.

Warren White's Example

In 1971, Warren White constructed another, strikingly counter-intuitive example.

Theorem (W. White, 1971)

There exists an Anosov structure for the translation $(x, y) \mapsto (x + 1, y)$.

White's Construction (sketch)

White constructs the foliations by integrating an orthonormal pair of vector fields $\vec{e}_s$, $\vec{e}_u$ which are independent of the second coordinate

White's Construction (sketch)

White constructs the foliations by integrating an orthonormal pair of vector fields $\vec{e}_s$, $\vec{e}_u$ which are independent of the second coordinate but vary periodically with the first coordinate so as to perform a full rotation as the first coordinate varies over a unit interval;

White's Construction (sketch)

White constructs the foliations by integrating an orthonormal pair of vector fields $\vec{e}_s$, $\vec{e}_u$ which are independent of the second coordinate but vary periodically with the first coordinate so as to perform a full rotation as the first coordinate varies over a unit interval; he then distorts the Euclidean metric by decreeing the length of $\vec{e}_u$ (resp. $\vec{e}_s$) at (x,y) to be λ^x (resp λ^{-x}) times its Euclidean length, for some $\lambda > 1$.

White's Construction (sketch)

White constructs the foliations by integrating an orthonormal pair of vector fields $\vec{e}_s$, $\vec{e}_u$ which are independent of the second coordinate but vary periodically with the first coordinate so as to perform a full rotation as the first coordinate varies over a unit interval; he then distorts the Euclidean metric by decreeing the length of $\vec{e}_u$ (resp. $\vec{e}_s$) at (x,y) to be λ^x (resp λ^{-x}) times its Euclidean length, for some $\lambda > 1$.
This automatically gives the hyperbolic estimates, and by further controlling the rotation of the vector fields with x the metric can be made complete.

Pedro Mendes' Conjecture

In 1977, Pedro Mendes studied general properties of Anosov diffeomorphisms in $\mathbb{R}^2$,

[1] "Prolongation" will be defined later.

Pedro Mendes' Conjecture

In 1977, Pedro Mendes studied general properties of Anosov diffeomorphisms in $\mathbb{R}^2$, proving that such a diffeomorphism has at most one non-wandering point (which of course must then be a fixed point)

[1] "Prolongation" will be defined later.

Pedro Mendes' Conjecture

In 1977, Pedro Mendes studied general properties of Anosov diffeomorphisms in $\mathbb{R}^2$, proving that such a diffeomorphism has at most one non-wandering point (which of course must then be a fixed point) and that every wandering point has empty prolongation.[1]

[1] "Prolongation" will be defined later.

Pedro Mendes' Conjecture

In 1977, Pedro Mendes studied general properties of Anosov diffeomorphisms in $\mathbb{R}^2$, proving that such a diffeomorphism has at most one non-wandering point (which of course must then be a fixed point) and that every wandering point has empty prolongation.[1] Based on these results, Mendes conjectured

[1] "Prolongation" will be defined later.

Pedro Mendes' Conjecture

In 1977, Pedro Mendes studied general properties of Anosov diffeomorphisms in $\mathbb{R}^2$, proving that such a diffeomorphism has at most one non-wandering point (which of course must then be a fixed point) and that every wandering point has empty prolongation.[1] Based on these results, Mendes conjectured that White's example is (up to topological conjugacy) the only example of an Anosov diffeomorphism in $\mathbb{R}^2$ other than the linear hyperbolic example.

[1] "Prolongation" will be defined later.

Pedro Mendes' Conjecture

In 1977, Pedro Mendes studied general properties of Anosov diffeomorphisms in $\mathbb{R}^2$, proving that such a diffeomorphism has at most one non-wandering point (which of course must then be a fixed point) and that every wandering point has empty prolongation.[1] Based on these results, Mendes conjectured that White's example is (up to topological conjugacy) the only example of an Anosov diffeomorphism in $\mathbb{R}^2$ other than the linear hyperbolic example.

Mendes' Conjecture

Every Anosov diffeomorphism of $\mathbb{R}^2$ is topologically conjugate either to a linear hyperbolic map or to a translation.

[1] "Prolongation" will be defined later.

Foliated Conjugacy

Two years ago, Jorge Groisman and I, visiting UAB (hosted by Lluis), shared an office and fell into conversation about Mendes' conjecture.

Foliated Conjugacy

Two years ago, Jorge Groisman and I, visiting UAB (hosted by Lluis), shared an office and fell into conversation about Mendes' conjecture. We attempted to prove the conjecture by taking advantage of the two foliations.

Foliated Conjugacy

Two years ago, Jorge Groisman and I, visiting UAB (hosted by Lluis), shared an office and fell into conversation about Mendes' conjecture. We attempted to prove the conjecture by taking advantage of the two foliations. We failed.

Foliated Conjugacy

But in the process we stumbled upon a finer, but natural structure which has provided a rich family of examples.

Foliated Conjugacy

But in the process we stumbled upon a finer, but natural structure which has provided a rich family of examples.

Definition: Foliated Conjugacy

A **foliated conjugacy** between two Anosov diffeomorphisms $f, g : \mathbb{R}^2 \to \mathbb{R}^2$ is a homeomorphism $h : \mathbb{R}^2 \to \mathbb{R}^2$ which

- conjugates f with g: $g \circ h = h \circ f$
- maps leaves of the stable (resp. unstable) foliation for f to leaves of the corresponding foliation for g.

We call f and g **equivalent** if there is a foliated conjugacy between them.

Two Kinds of Invariant Open Discs

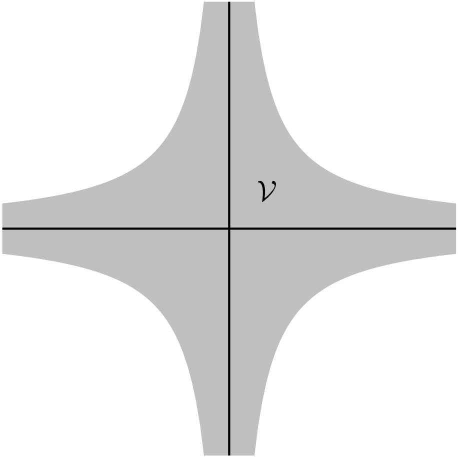
Our examples are all built from the restriction of the linear hyperbolic map $f(x, y) = (2x, y/2)$ to an invariant topological disc.

Two Kinds of Invariant Open Discs

Our examples are all built from the restriction of the linear hyperbolic map $f(x, y) = (2x, y/2)$ to an invariant topological disc. Note that the function $\tau(x, y) = xy$ is invariant under f $(\tau \circ f = \tau)$, and in particular we can form two kinds of f-invariant open topological discs:

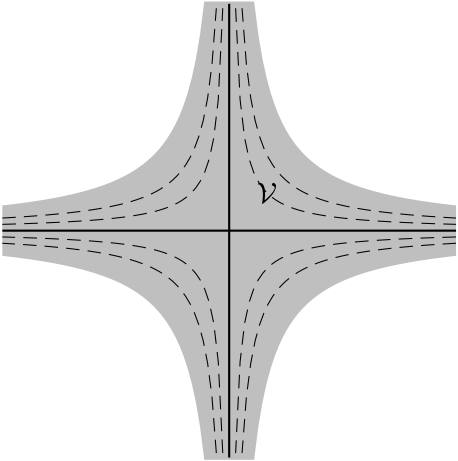
Invariant Open Discs Containing a Fixed Point

Given $a < 0 < b$, the set $\mathcal{V} = \{(x, y) | a < \tau(x, y) < b\}$ is an $f-invariant$ open neighborhood of the origin (the unique fixed point of f) homeomorphic to an open disc:



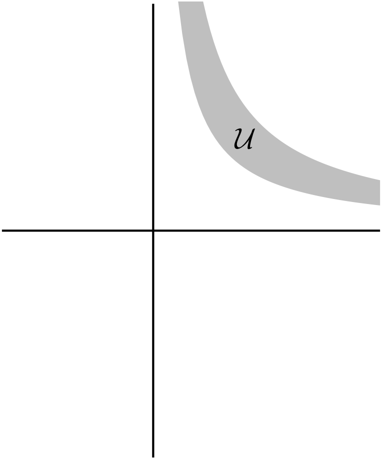
Invariant Open Discs Containing a Fixed Point

Given $a < 0 < b$, the set $\mathcal{V} = \{(x, y) | a < \tau(x, y) < b\}$ is an *f − invariant* open neighborhood of the origin (the unique fixed point of f) homeomorphic to an open disc: note that the level curves of τ give a foliation of $\mathcal{V}$ by f-invariant curves.



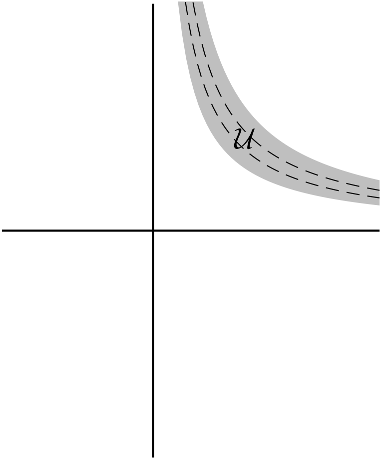
Invariant Open Discs Containing No Fixed Point

Given $0 \leq a < b$, the set $\mathcal{U} = \{(x, y) | a < \tau(x, y) < b, \quad x > 0\}$ is an $f - invariant$ open set *not* containing the origin, and homeomorphic to an open disc;



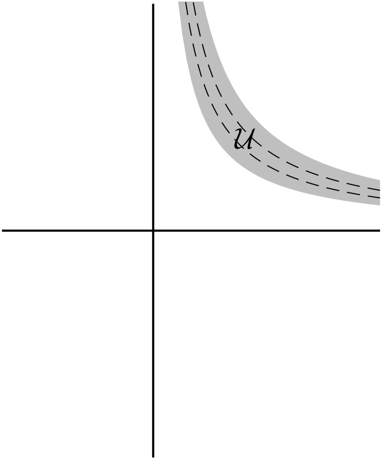
Invariant Open Discs Containing No Fixed Point

Given $0 \leq a < b$, the set $\mathcal{U} = \{(x, y) | a < \tau(x, y) < b, \quad x > 0\}$ is an $f-invariant$ open set *not* containing the origin, and homeomorphic to an open disc; again, the level curves of τ give a foliation of $\mathcal{U}$ by f-invariant curves.



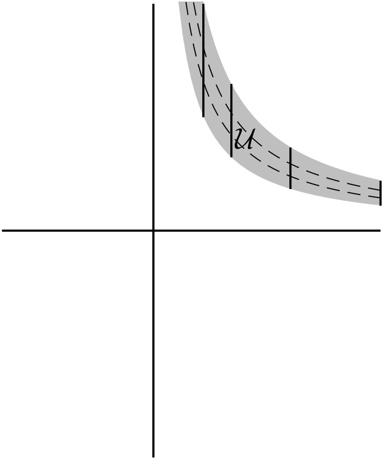
Invariant Open Discs Containing No Fixed Point

It is easy to see that the restriction of f to $\mathcal{U}$ is conjugate to a translation:



Invariant Open Discs Containing No Fixed Point

It is easy to see that the restriction of f to $\mathcal{U}$ is conjugate to a translation: for example the vertical lines $x = 2^k$, $k = 1, 2, \ldots$ cut $\mathcal{U}$ into open sets, each mapped to the next.



Constructing a Complete Metric on $\mathcal{U}$

Given $a < b$, consider the function

$$\varphi(t) = \frac{1}{(t-a)(b-t)}$$

on the interval (a, b). It is positive, unimodal, and for any $c \in (a, b)$ both of the (improper) integrals

$$\int_a^c \varphi(t)\ dt \text{ and } \int_c^b \varphi(t)\ dt$$

diverge.

Constructing a Complete Metric on $\mathcal{U}$

To define a Riemann metric on $\mathcal{U}$ we multiply the Euclidean length of every vector at a point by the value of $g = \varphi \circ \tau$ there.

Constructing a Complete Metric on $\mathcal{U}$

To define a Riemann metric on $\mathcal{U}$ we multiply the Euclidean length of every vector at a point by the value of $g = \varphi \circ \tau$ there. The fact that g is f-invariant means that the stretching and shrinking by f with respect to the Euclidean metric,

Constructing a Complete Metric on $\mathcal{U}$

To define a Riemann metric on $\mathcal{U}$ we multiply the Euclidean length of every vector at a point by the value of $g = \varphi \circ \tau$ there. The fact that g is f-invariant means that the stretching and shrinking by f with respect to the Euclidean metric, and the fact that the vertical (resp. horizontal) lines are the stable (resp. unstable) foliation,

Constructing a Complete Metric on $\mathcal{U}$

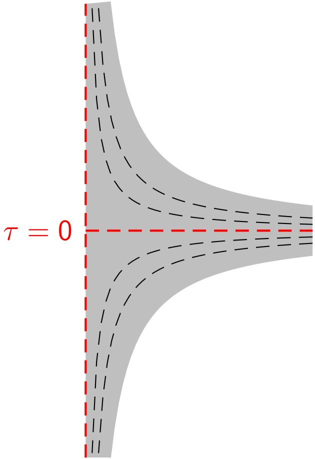
To define a Riemann metric on $\mathcal{U}$ we multiply the Euclidean length of every vector at a point by the value of $g = \varphi \circ \tau$ there. The fact that g is f-invariant means that the stretching and shrinking by f with respect to the Euclidean metric, and the fact that the vertical (resp. horizontal) lines are the stable (resp. unstable) foliation, remains in the new metric.

Constructing a Complete Metric on $\mathcal{U}$

To define a Riemann metric on $\mathcal{U}$ we multiply the Euclidean length of every vector at a point by the value of $g = \varphi \circ \tau$ there. The fact that g is f-invariant means that the stretching and shrinking by f with respect to the Euclidean metric, and the fact that the vertical (resp. horizontal) lines are the stable (resp. unstable) foliation, remains in the new metric.
The completeness of the metric follows (with some work) from the divergence of integrals of φ which involve a or b.

Complete Metrics

Note that this construction cannot be carried out for the analogue of $\mathcal{U}$ with $a < 0 < b$.



Transfer to $\mathbb{R}^2$

Now, the Riemann Mapping Theorem gives us a diffeomorphism between $\mathcal{U}$ and $\mathbb{R}^2$.

Transfer to $\mathbb{R}^2$

Now, the Riemann Mapping Theorem gives us a diffeomorphism between $\mathcal{U}$ and $\mathbb{R}^2$. Conjugating our example by this diffeomorphism, we obtain an Anosov structure for a diffeomorphism of the plane which is (conjugate to) a translation.

Transfer to $\mathbb{R}^2$

Now, the Riemann Mapping Theorem gives us a diffeomorphism between $\mathcal{U}$ and $\mathbb{R}^2$. Conjugating our example by this diffeomorphism, we obtain an Anosov structure for a diffeomorphism of the plane which is (conjugate to) a translation. A similar construction can be carried out replacing $\mathcal{U}$ with $\mathcal{V}$, to get an Anosov diffeomorphism of $\mathbb{R}^2$ with a single fixed point.

Transfer to $\mathbb{R}^2$

Now, the Riemann Mapping Theorem gives us a diffeomorphism between $\mathcal{U}$ and $\mathbb{R}^2$. Conjugating our example by this diffeomorphism, we obtain an Anosov structure for a diffeomorphism of the plane which is (conjugate to) a translation. A similar construction can be carried out replacing $\mathcal{U}$ with $\mathcal{V}$, to get an Anosov diffeomorphism of $\mathbb{R}^2$ with a single fixed point. The point here is that we can construct a variety of non-equivalent examples by using different f-invariant open discs and then transferring to $\mathbb{R}^2$ via the Riemann mapping theorem.

Non-equivalent examples

We will discuss two invariants of the equivalence relation, both based on the homeomorphism type of the two foliations.

Accessibility

The first involves the connection between points using leaves of the foliation.

Accessibility

The first involves the connection between points using leaves of the foliation.

Definition: Accessibility

Given the pair of foliations $\mathcal{F}^s$ and $\mathcal{F}^u$ coming from an Anosov structure, we say that $q \in \mathbb{R}^2$ is ***n*-accessible** from $p \in \mathbb{R}^2$

Accessibility

The first involves the connection between points using leaves of the foliation.

Definition: Accessibility

Given the pair of foliations $\mathcal{F}^s$ and $\mathcal{F}^u$ coming from an Anosov structure, we say that $q \in \mathbb{R}^2$ is ***n*-accessible** from $p \in \mathbb{R}^2$ if there exist points $p = p_0, p_1, \ldots, p_n = q$

Accessibility

The first involves the connection between points using leaves of the foliation.

Definition: Accessibility

Given the pair of foliations $\mathcal{F}^s$ and $\mathcal{F}^u$ coming from an Anosov structure, we say that $q \in \mathbb{R}^2$ is ***n*-accessible** from $p \in \mathbb{R}^2$ if there exist points $p = p_0, p_1, \ldots, p_n = q$ such that each successive pair p_i, p_{i+1}

Accessibility

The first involves the connection between points using leaves of the foliation.

Definition: Accessibility

Given the pair of foliations $\mathcal{F}^s$ and $\mathcal{F}^u$ coming from an Anosov structure, we say that $q \in \mathbb{R}^2$ is ***n*-accessible** from $p \in \mathbb{R}^2$ if there exist points $p = p_0, p_1, \ldots, p_n = q$ such that each successive pair p_i, p_{i+1} lies on a common stable or unstable leaf.

We say that p and q are *n*-connected in this case.

Accessibility

Using the product structure of $\mathcal{F}^s$ and $\mathcal{F}^u$ and the connectedness of the plane, it is easy to show that any two points $p, q \in \mathbb{R}^2$ are n-connected for some finite n.

Accessibility

Using the product structure of $\mathcal{F}^s$ and $\mathcal{F}^u$ and the connectedness of the plane, it is easy to show that any two points $p, q \in \mathbb{R}^2$ are n-connected for some finite n. For each pair $p, q \in \mathbb{R}^2$, we minimize the degree of accessibility n:

Definition: $\mathcal{N}(p, q)$

$$\mathcal{N}(p, q) := \min\{n | p \text{ and } q \text{ are } n\text{-connected}\}.$$

Degree of Inaccessibility

By maximizing over all pairs of points, we obtain an invariant of foliated conjugacy, which we call the **degree of inaccessibility** of the Anosov structure.

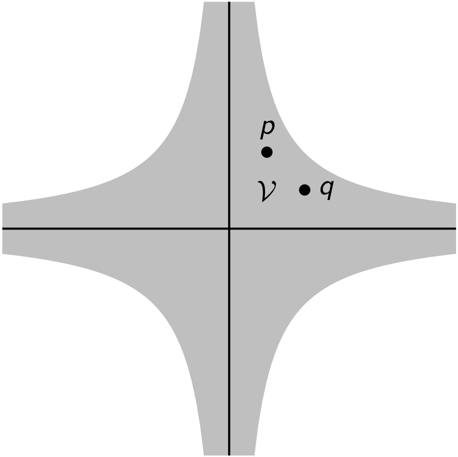
Definition: Degree of Inaccessibility

The **degree of inaccessibility** for the pair of foliations $\mathcal{F}^s$ and $\mathcal{F}^u$ is

$$\sup\{\mathcal{N}(p,q) | p, q \in \mathbb{R}^2\}.$$

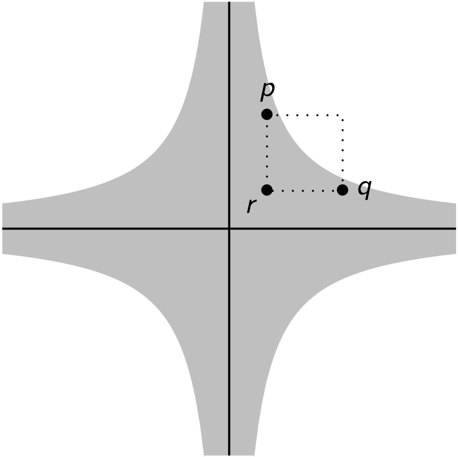
Accessibility in $\mathcal{V}$

For the foliations coming from our construction using the neighborhood $\mathcal{V}$, the degree of inaccessibility is 2:



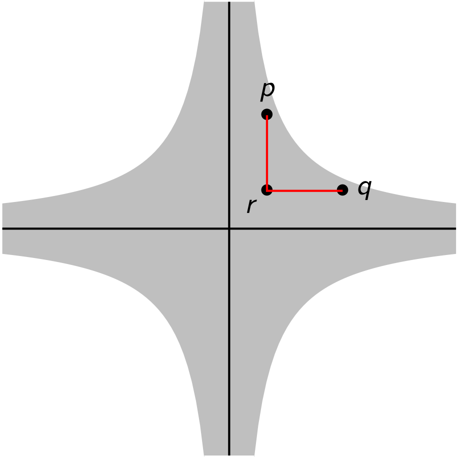
Accessibility in $\mathcal{V}$

For the foliations coming from our construction using the neighborhood $\mathcal{V}$, the degree of inaccessibility is 2: Given two points $p, q \in \mathcal{V}$, the rectangle with vertical and horizontal sides and p and q vertices has at least one vertex r in $\mathcal{V}$



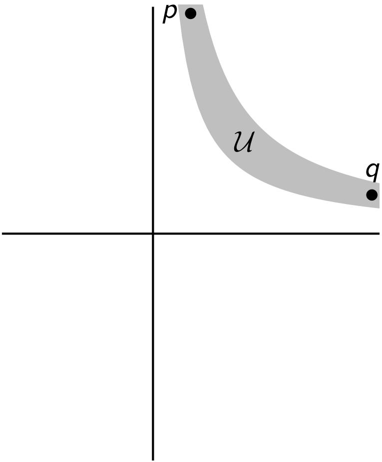
Accessibility in $\mathcal{V}$

For the foliations coming from our construction using the neighborhood $\mathcal{V}$, the degree of inaccessibility is 2: Given two points $p, q \in \mathcal{V}$, the rectangle with vertical and horizontal sides and p and q vertices has at least one vertex r in $\mathcal{V}$, and the triple p, r, q is a 2-connection.



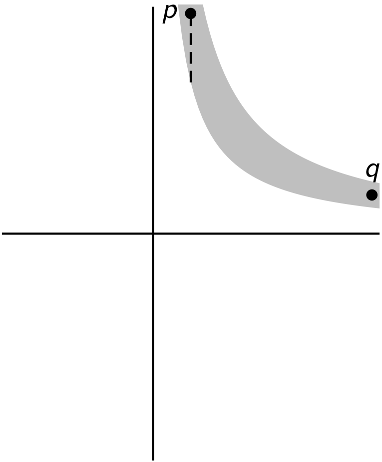
Infinite Degree of Inaccessibility in $\mathcal{U}$

By contrast, for $\mathcal{U}$ with $0 < a < b$, the degree of inaccessibility is infinite:



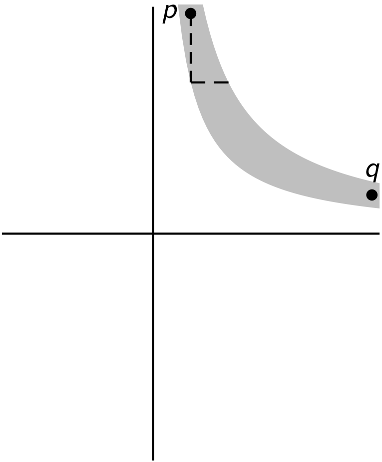
Infinite Degree of Inaccessibility in $\mathcal{U}$

By contrast, for $\mathcal{U}$ with $0 < a < b$, the degree of inaccessibility is infinite: from p we can only go as far down as the lower edge of $\mathcal{U}$,



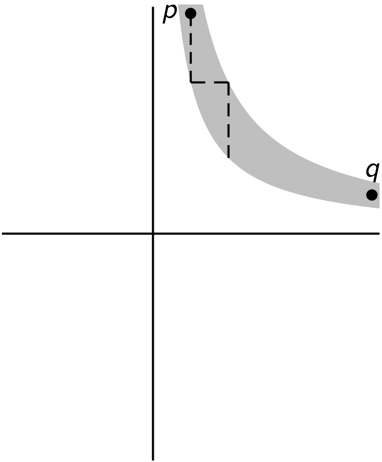
Infinite Degree of Inaccessibility in $\mathcal{U}$

By contrast, for $\mathcal{U}$ with $0 < a < b$, the degree of inaccessibility is infinite: from p we can only go as far down as the lower edge of $\mathcal{U}$, then from there only as far to the right as the upper edge of $\mathcal{U}$,



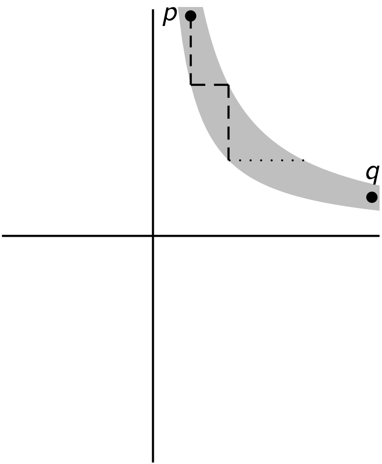
Infinite Degree of Inaccessibility in $\mathcal{U}$

By contrast, for $\mathcal{U}$ with $0 < a < b$, the degree of inaccessibility is infinite: from p we can only go as far down as the lower edge of $\mathcal{U}$, then from there only as far to the right as the upper edge of $\mathcal{U}$, ...and so on.



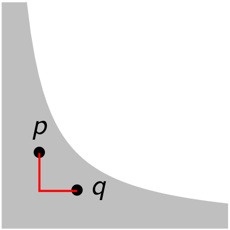
Infinite Degree of Inaccessibility in $\mathcal{U}$

By contrast, for $\mathcal{U}$ with $0 < a < b$, the degree of inaccessibility is infinite: from p we can only go as far down as the lower edge of $\mathcal{U}$, then from there only as far to the right as the upper edge of $\mathcal{U}$, ...and so on. So there exist pairs of points in $\mathcal{U}$ for which $\mathcal{N}(p, q)$ is arbitrarily high.



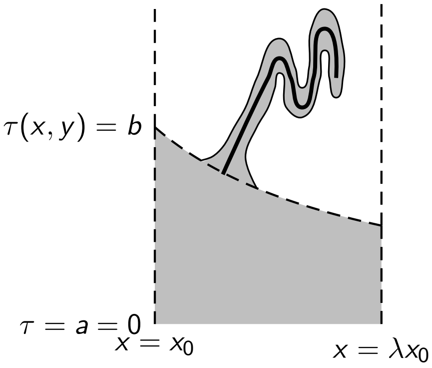
Finite Degree of Inaccessibility in $\mathcal{U}$

When $a = 0 < b$ the situation is similar to that in $\mathcal{V}$: the degree of accessibility is 2.



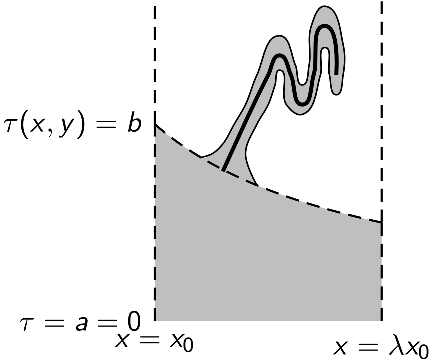
Arbitrary Finite Degree of Inaccessibility

It is also possible to create examples for which the degree of inaccessibility is equal to any finite value above 2.



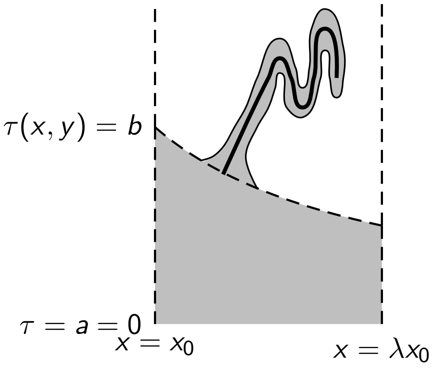
Arbitrary Finite Degree of Inaccessibility

It is also possible to create examples for which the degree of inaccessibility is equal to any finite value above 2. We can adjoin to one edge of $\mathcal{U}$ (with $a = 0 < b$) a “whisker” together with a neighborhood, contained in a fundamental neighborhood of f,



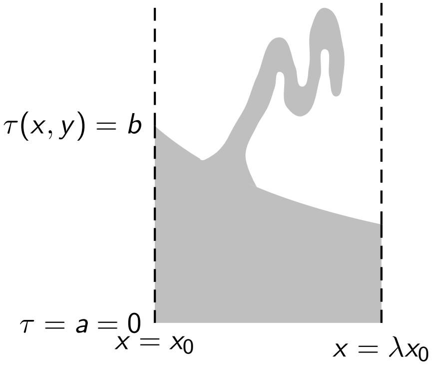
Arbitrary Finite Degree of Inaccessibility

It is also possible to create examples for which the degree of inaccessibility is equal to any finite value above 2. We can adjoin to one edge of $\mathcal{U}$ (with $a = 0 < b$) a "whisker" together with a neighborhood, contained in a fundamental neighborhood of f, then use f to copy it on the edge of $\mathcal{U}$ in every fundamental neighborhood of f.



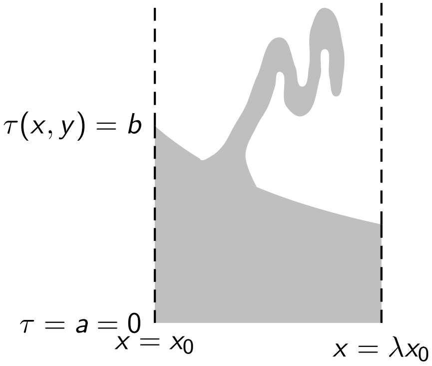
Arbitrary Finite Degree of Inaccessibility

The new open disc is diffeomorphic to $\mathcal{U}$ and we can use this to modify τ so as to get an f-invariant function on this new set, and then mimic the construction of a complete metric on this disc.



Arbitrary Finite Degree of Inaccessibility

The new open disc is diffeomorphic to $\mathcal{U}$ and we can use this to modify τ so as to get an f-invariant function on this new set, and then mimic the construction of a complete metric on this disc. The "wiggles" in the whisker increase the degree of inaccessibility by as much as we want.



Are All Examples Given by Our Construction?

The question naturally arises, is every example of an Anosov structure in the plane equivalent to one constructed from restriction to an invariant open disc for the linear hyperbolic map?

Are All Examples Given by Our Construction?

The question naturally arises, is every example of an Anosov structure in the plane equivalent to one constructed from restriction to an invariant open disc for the linear hyperbolic map? The question more or less rests on whether we can take the two foliations $\mathcal{F}^s$ and $\mathcal{F}^u$ to the foliations by vertical (resp. horizontal) lines via some homeomorphism from $\mathbb{R}^2$ to an open disc in $\mathbb{R}^2$.

Are All Examples Given by Our Construction?

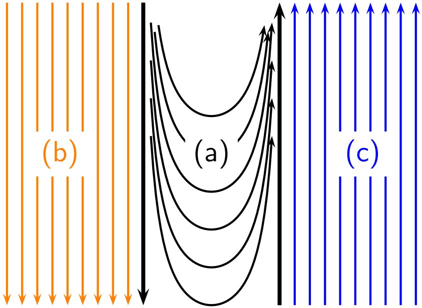
The question naturally arises, is every example of an Anosov structure in the plane equivalent to one constructed from restriction to an invariant open disc for the linear hyperbolic map? The question more or less rests on whether we can take the two foliations $\mathcal{F}^s$ and $\mathcal{F}^u$ to the foliations by vertical (resp. horizontal) lines via some homeomorphism from $\mathbb{R}^2$ to an open disc in $\mathbb{R}^2$.

Definition: Quasi-Parallel Foliation

A foliation $\mathcal{F}$ of $\mathbb{R}^2$ is **quasi-parallel** if there exists a homeomorphism from $\mathbb{R}^2$ to some open topological disc taking the leaves of $\mathcal{F}$ to horizontal (or equivalently, vertical) lines.

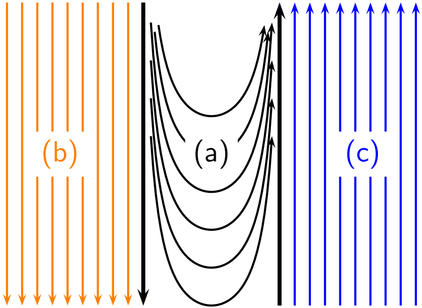
Quasi-Parallel vs Parallelizable Foliations

This is not the same as *parallelizability*, that is, existence of a homeomorphism of the plane to itself taking leaves to horizontal lines, or equivalently, the existence of a cross-section (a curve crossing every leaf transversally): a well known obstruction to parallelizability is the presence of a *Reeb component*.



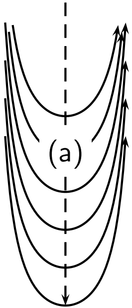
Quasi-Parallel vs Parallelizable Foliations

No cross-section can join the two vertical leaves at the edge of the Reeb component (the region marked (a)), so the foliation is not parallelizable.



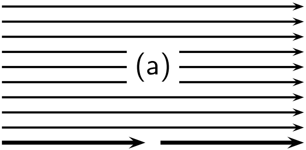
Quasi-Parallel Foliations with Reeb Components

However, the dashed vertical line down the middle of the Reeb component intersects every leaf interior to the Reeb component, so the restriction of the foliation to this open strip *is* parallelizable.



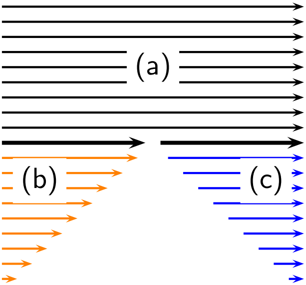
Quasi-Parallel Foliations with Reeb Components

By mapping this cross-section to the open interval $\{0\} \times (0, 1)$, we can clearly find a homeomorphism taking leaves of the Reeb component to horizontal lines in the open square $(-1, 1) \times (0, 1)$ and the two edges of this component to the open intervals $(-1, 0) \times \{0\}$ and $(0, 1) \times \{0\}$.



Quasi-Parallel Foliations with Reeb Components

We can then extend this homeomorphism so as to take the regions marked (b) and (c) (each of which is individually parallelizable) into open triangles abutting these two segments.



An Obstruction to Quasi-Parallelizability

However, there is an obstruction to quasi-parallizability, which can be formulated using the fact that all foliations of $\mathbb{R}^2$ by lines are orientable, and so can be regarded as integral curves of a flow.

Prolongation

We formulate this in terms of the **prolongational limit** relation, studied extensively by Joe Auslander and Peter Seibert.

Prolongation

We formulate this in terms of the **prolongational limit** relation, studied extensively by Joe Auslander and Peter Seibert.

Definition: Prolongation

The point y is in the **forward prolongation** of the point x, denoted $\boldsymbol{y \in J_+(x)}$
(and x is in the **backward prolongation of** y, $\boldsymbol{x \in J_-(y)}$)
under the dynamical system φ^t,

Prolongation

We formulate this in terms of the **prolongational limit** relation, studied extensively by Joe Auslander and Peter Seibert.

Definition: Prolongation

The point y is in the **forward prolongation** of the point x, denoted $\boldsymbol{y \in J_+(x)}$
(and x is in the **backward prolongation of** y, $\boldsymbol{x \in J_-(y)}$)
under the dynamical system φ^t,
if there exists a sequence of points $x_k \to x$ and a sequence of times $t_k \to \infty$ so that

$$y_k = \varphi^{t_k}(x_k) \to y.$$

Prolongation

We formulate this in terms of the **prolongational limit** relation, studied extensively by Joe Auslander and Peter Seibert.

Definition: Prolongation

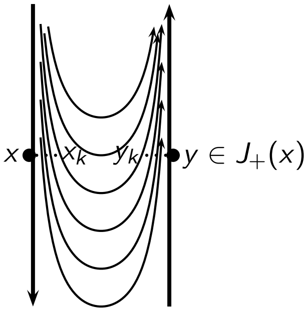
The point y is in the **forward prolongation** of the point x, denoted $\boldsymbol{y \in J_+(x)}$
(and x is in the **backward prolongation of** y, $\boldsymbol{x \in J_-(y)}$)
under the dynamical system φ^t,
if there exists a sequence of points $x_k \to x$ and a sequence of times $t_k \to \infty$ so that

$$y_k = \varphi^{t_k}(x_k) \to y.$$

(When $x = y$, this is precisely the definition of non-wandering.)

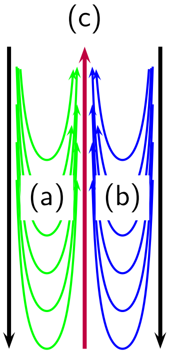
Prolongation

One way to picture prolongation, in the case of a planar flow, is as a Reeb component.



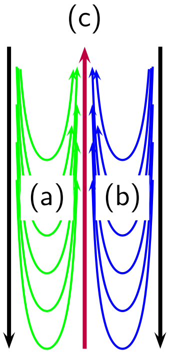
An Obstruction to Quasi-Parallelizability

Consider the situation of two Reeb components, with the interior leaves in each curling up, and separated by a single orbit.



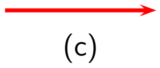
An Obstruction to Quasi-Parallelizability

We claim this cannot be part of a quasi-parallel foliation.



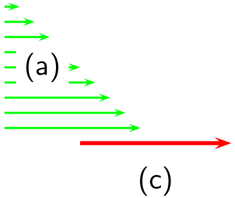
An Obsruction to Quasi-Parallelizability

To see this, note that in a quasi-parallelized picture, the horizontal lines must all be oriented in the same direction, which we have taken to be left-to-right. The orbit (c) separating the two Reeb components maps to an open interval $I = (\alpha, \beta)$, which we can take on the x-axis.



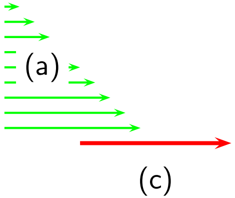
An Obsruction to Quasi-Parallelizability

Since the orbit on the left edge of (a) is the *backward* prolongational limit of these points, the orbits in (a) map to line segments extending to the left of α.



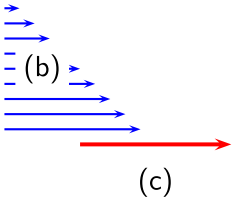
An Obsruction to Quasi-Parallelizability

Since orbits in (a) see (c) on their *right* side, the image of (a) is in the upper half plane.



An Obsruction to Quasi-Parallelizability

However, (c) *also* has the right edge of (b) in its backward prolongational limit, so the orbits of (b) must also extend to the left of the image of (c), and the image of (b) must be in the upper half plane.



An Obsruction to Quasi-Parallelizability

It follows that the images of (a) and (b) must intersect, a contradiction to the fact that these are images under a homeomorphism of the whole plane into the plane.

An Obsruction to Quasi-Parallelizability

It follows that the images of (a) and (b) must intersect, a contradiction to the fact that these are images under a homeomorphism of the whole plane into the plane. The critical situation here is that two Reeb components curling in the same direction are separated by a single orbit.

An Obsruction to Quasi-Parallelizability

It follows that the images of (a) and (b) must intersect, a contradiction to the fact that these are images under a homeomorphism of the whole plane into the plane. The critical situation here is that two Reeb components curling in the same direction are separated by a single orbit.

This can be formulated as: there is a leaf which separates two distinct leaves in its backward prolongation (or both in its forward prolongation).

An Obsruction to Quasi-Parallelizability

It follows that the images of (a) and (b) must intersect, a contradiction to the fact that these are images under a homeomorphism of the whole plane into the plane. The critical situation here is that two Reeb components curling in the same direction are separated by a single orbit.

This can be formulated as: there is a leaf which separates two distinct leaves in its backward prolongation (or both in its forward prolongation).

It is possible to adapt White's construction to exhibit an Anosov structure on $\mathbb{R}^2$ which exhibits this phenomenon, and hence is not given by one of our examples coming from an invariant open disc for f.

An Obsruction to Quasi-Parallelizability

It follows that the images of (a) and (b) must intersect, a contradiction to the fact that these are images under a homeomorphism of the whole plane into the plane. The critical situation here is that two Reeb components curling in the same direction are separated by a single orbit.

This can be formulated as: there is a leaf which separates two distinct leaves in its backward prolongation (or both in its forward prolongation).

It is possible to adapt White's construction to exhibit an Anosov structure on $\mathbb{R}^2$ which exhibits this phenomenon, and hence is not given by one of our examples coming from an invariant open disc for f. We have not determined whether this is true of White's original example.

Summary

- Mendes' conjecture remains open: Are there Anosov structures for diffeomorphisms of $\mathbb{R}^2$ which are not conjugate to either linear hyperbolic maps or translations?

Summary

- Mendes' conjecture remains open: Are there Anosov structures for diffeomorphisms of $\mathbb{R}^2$ which are not conjugate to either linear hyperbolic maps or translations?
- Examples of Anosov diffeomorphisms of $\mathbb{R}^2$ in either topological conjugacy class can be constructed by restricting a linear hyperbolic map to an invariant topological disc and transferring to $\mathbb{R}^2$ via the Riemann mapping theorem.

Summary

- Mendes' conjecture remains open: Are there Anosov structures for diffeomorphisms of $\mathbb{R}^2$ which are not conjugate to either linear hyperbolic maps or translations?
- Examples of Anosov diffeomorphisms of $\mathbb{R}^2$ in either topological conjugacy class can be constructed by restricting a linear hyperbolic map to an invariant topological disc and transferring to $\mathbb{R}^2$ via the Riemann mapping theorem.
- These include infinitely many mutually non-equivalent Anosov diffeomorphisms of $\mathbb{R}^2$.

Summary

- Mendes' conjecture remains open: Are there Anosov structures for diffeomorphisms of $\mathbb{R}^2$ which are not conjugate to either linear hyperbolic maps or translations?
- Examples of Anosov diffeomorphisms of $\mathbb{R}^2$ in either topological conjugacy class can be constructed by restricting a linear hyperbolic map to an invariant topological disc and transferring to $\mathbb{R}^2$ via the Riemann mapping theorem.
- These include infinitely many mutually non-equivalent Anosov diffeomorphisms of $\mathbb{R}^2$.
- There are Anosov structures (in either topological conjgacy class) which are not equivalent to the restriction of a linear hyperbolic map to an invariant open topological disc.

Thanks

Gracias por su atención!

References

1. Pedro Mendes, *On Anosov diffeomorphisms on the plane*, PROCEEDINGS OF AMS (1977) pp. 231-235
2. Jorge Groisman & Zbigniew Nitecki, *Foliations and Conjugacy: Anosov structures in the plane* ERGODIC THEORY & DYNAMICAL SYSTEMS (2014)
3. Warren White, *An Anosov translation*, in M. Peixoto (ed.) DYNAMICAL SYSTEMS *Proceedings of a Symposium held at the Unversity of Bahia, Salvador, Brasil, July 26-August 14, 1971* 1971, pp. 667-670