LOOPS OF TRANSITIVE INTERVAL MAPS

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In the paper

S. Kolyada, M. M. and L. Snoha, *Spaces of transitive interval maps*, Ergod. Th. & Dynam. Sys., posted electronically on August 5, 2014

we were investigating topology of various spaces of continuous transitive interval maps. Let us denote by $\mathcal{T}_n$ the space of all continuous transitive piecewise monotone interval maps of modality n. For each $n \geq 1$ we found a loop in $\mathcal{T}_n \cup \mathcal{T}_{n+1}$, which is not contractible in $\mathcal{T}_n \cup \mathcal{T}_{n+1}$. We left as an open problem the question whether this loop can be contracted in some larger space obtained by adding spaces $\mathcal{T}_m$ for one or more m.

Here we solve this problem and investigate topology (and, in a sense, geometry) of the spaces of transitive interval maps with constant slope and given modality.

Two basic questions that we encounter are:

1) How to recognize that an interval map is transitive?

2) How to produce arcs in the spaces of transitive interval maps?

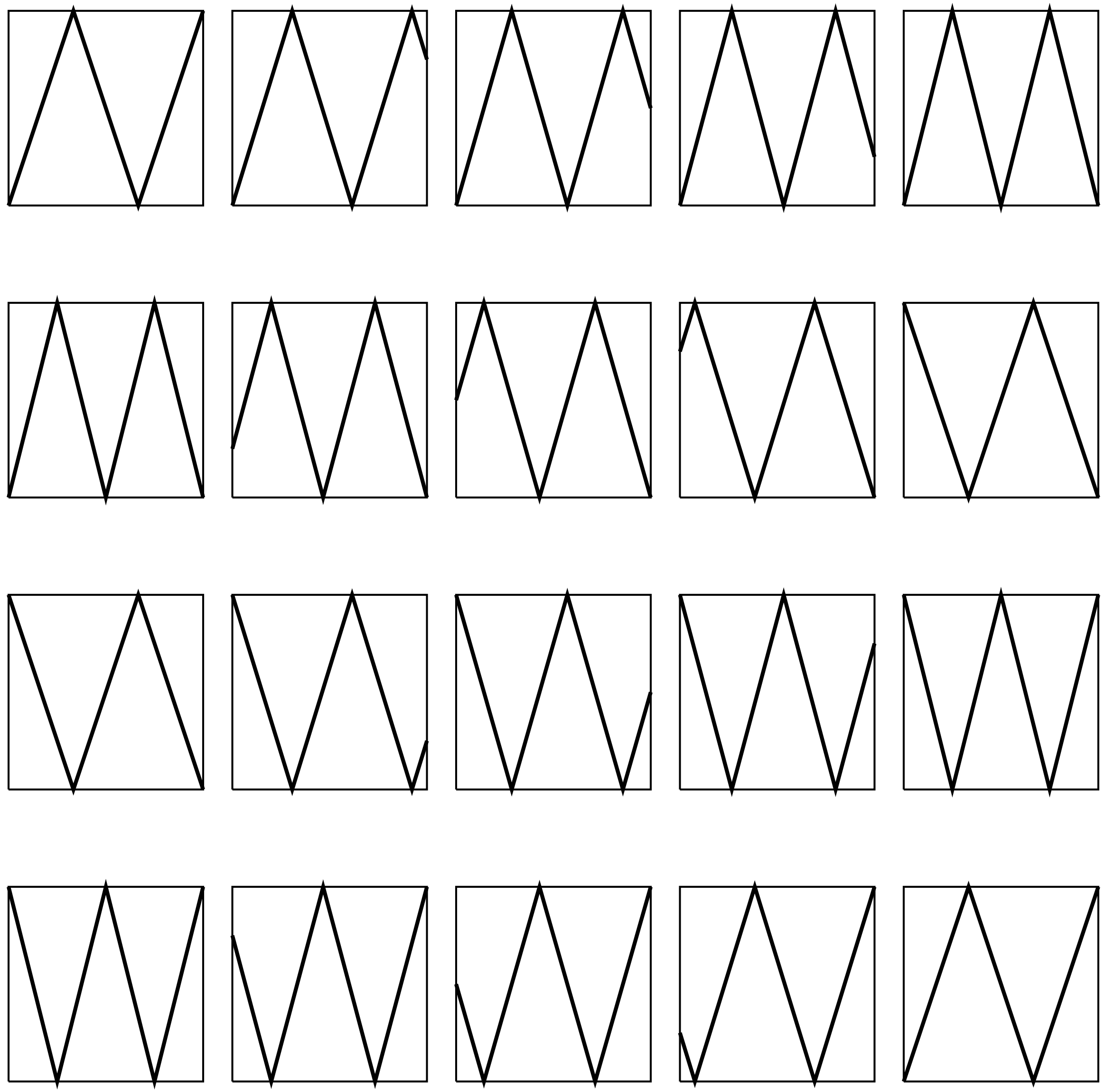
We work with maps of constant slope. We prove several lemmas that often help us to answer the first question. Then we introduce parametrization of our maps that helps us to deal with the second question.

We denote by $\mathcal{CS}_n$ the space of all piecewise linear maps of the interval $[0,1]$ to itself, with constant slope and of modality n. The set of all transitive maps from $\mathcal{CS}_n$ will be denoted by $\mathcal{TCS}_n$.

All those spaces are considered with the C^0 metric:

$$d(f,g) = \sup_{x\in[0,1]} |f(x) - g(x)|.$$

The loop in $\mathcal{T}_n \cup \mathcal{T}_{n+1}$, not contractible in $\mathcal{T}_n \cup \mathcal{T}_{n+1}$, that we mentioned earlier, is contained in $\mathcal{TCS}_n \cup \mathcal{TCS}_{n+1}$. We will call it *the basic loop of order* n. It looks as follows (here $n = 2$):



The main result:

Theorem 1. *For every $n \geq 2$, the basic loops of order n and $n+1$ can be contracted in $\mathcal{TCS}_n \cup \mathcal{TCS}_{n+1} \cup \mathcal{TCS}_{n+2}$. Moreover, the basic loop of order 1 can be contracted in $\mathcal{TCS}_1 \cup \mathcal{TCS}_2 \cup \mathcal{TCS}_4$.*

The situation is similar as for the following model. Think about the sequence of spaces $\mathbb{R}^n$, $n = 0, 1, 2, \ldots$, where each space is a subset of the next one. Set $R_n = \mathbb{R}^n \setminus \mathbb{R}^{n-1}$ for $n = 1, 2, 3, \ldots$. Then the fundamental group of the space

$$R_n \cup R_{n+1} = \mathbb{R}^{n+1} \setminus \mathbb{R}^{n-1} = (\mathbb{R}^2 \setminus \{0\}) \times \mathbb{R}^{n-1}$$

is nontrivial, while the fundamental group of the space

$$R_n \cup R_{n+1} \cup R_{n+2} = \mathbb{R}^{n+2} \setminus \mathbb{R}^{n-1} = (\mathbb{R}^3 \setminus \{0\}) \times \mathbb{R}^{n-1}$$

is trivial.

To state the next result, we need a coding of the elements of $\mathcal{CS}_n$. The code is a sequence of values of the map at the endpoints and turning points. This gives us a sequence of length $n+2$. However, when we consider a union of spaces $\mathcal{CS}_n$ for several ns, we use the common length of the code, that is, the largest of the ns plus 2. Then the codes for some maps may be not unique. For instance, when we code the usual tent map using sequences of length 4 (instead of length 3, when the code would be $(0,1,0)$), the code may be $(0,0,1,0)$ or $(0,1,0,0)$. We do not use the code $(0,1,1,0)$, because we want the increasing and decreasing laps to alternate.

We need simple properties of coding.

Lemma 2. *The slope of a map* $f \in \bigcup_{i=1}^{n} \mathcal{CS}_i$ *with the code* $(a_0, a_1, \ldots, a_{n+1})$ *is* $\sum_{j=1}^{n+1} |a_j - a_{j-1}|$.

Lemma 3. *The map* $f \in \bigcup_{i=1}^{n} \mathcal{CS}_i$ *depends continuously on the parameters* $a_0, a_1, \ldots, a_{n+1}$ *(jointly).*

The space $\mathcal{TCS}_1$ is very simple.

Lemma 4. *A map $f \in \mathcal{CS}_1$ is transitive if and only if it has the code*

(1) $(a, 1, 0)$, *where* $a \in [0, 2 - \sqrt{2}]$, *or*

(2) $(1, 0, c)$, *where* $c \in [\sqrt{2} - 1, 1]$.

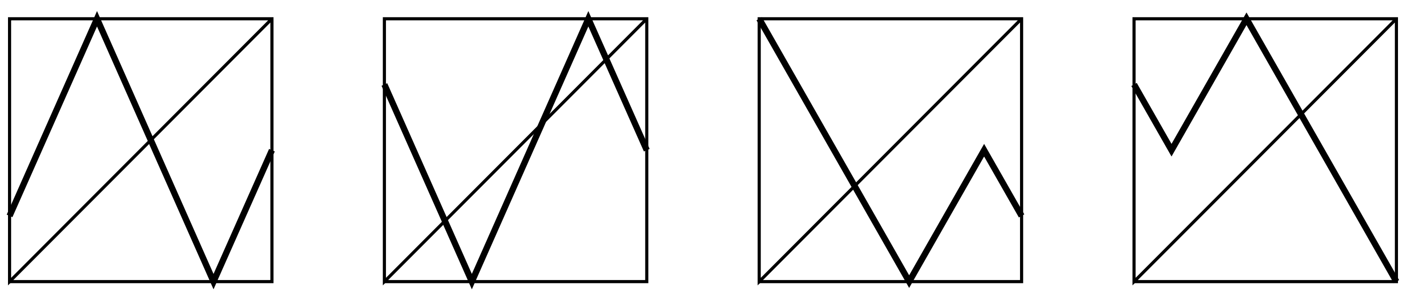
Lemma 5. *Let $f \in \mathcal{CS}_2$ be transitive. Then it has one of the four codes:*

(1) $(a, 1, 0, d)$, *where* $a \in [0, 1)$ *and* $d \in (0, 1]$;

(2) $(a, 0, 1, d)$, *where* $a \in (0, 1]$ *and* $d \in [0, 1)$;

(3) $(1, 0, c, d)$, *where* $c \in (0, 1]$ *and* $d \in [0, c)$;

(4) $(a, b, 1, 0)$, *where* $a \in (0, 1]$ *and* $b \in [0, a)$.



In the case $(a, 1, 0, d)$, if x is the fixed point on the second lap, then transitivity is equivalent to $a \leq x$ or $d \geq x$:

Lemma 6. *Let $f \in \mathcal{CS}_2$ have a code $(a, 1, 0, d)$ where $a \in [0, 1)$ and $d \in (0, 1]$. Then f is transitive if and only if*

$$d \leq a - 4 + \frac{2}{a} \qquad \textit{or} \qquad 1 - a \leq (1 - d) - 4 + \frac{2}{1 - d} \ . \tag{1}$$

In the case $(a, 0, 1, d)$, transitivity is equivalent to the slope being larger than 2:

Lemma 7. *Let $f \in \mathcal{CS}_2$ have a code $(a, 0, 1, d)$ where $a \in (0, 1]$ and $d \in [0, 1)$. Then f is transitive if and only if $a > d$.*

In the case $(1, 0, c, d)$, if the fixed point in the first lap is x, then transitivity is equivalent to $c \geq x$:

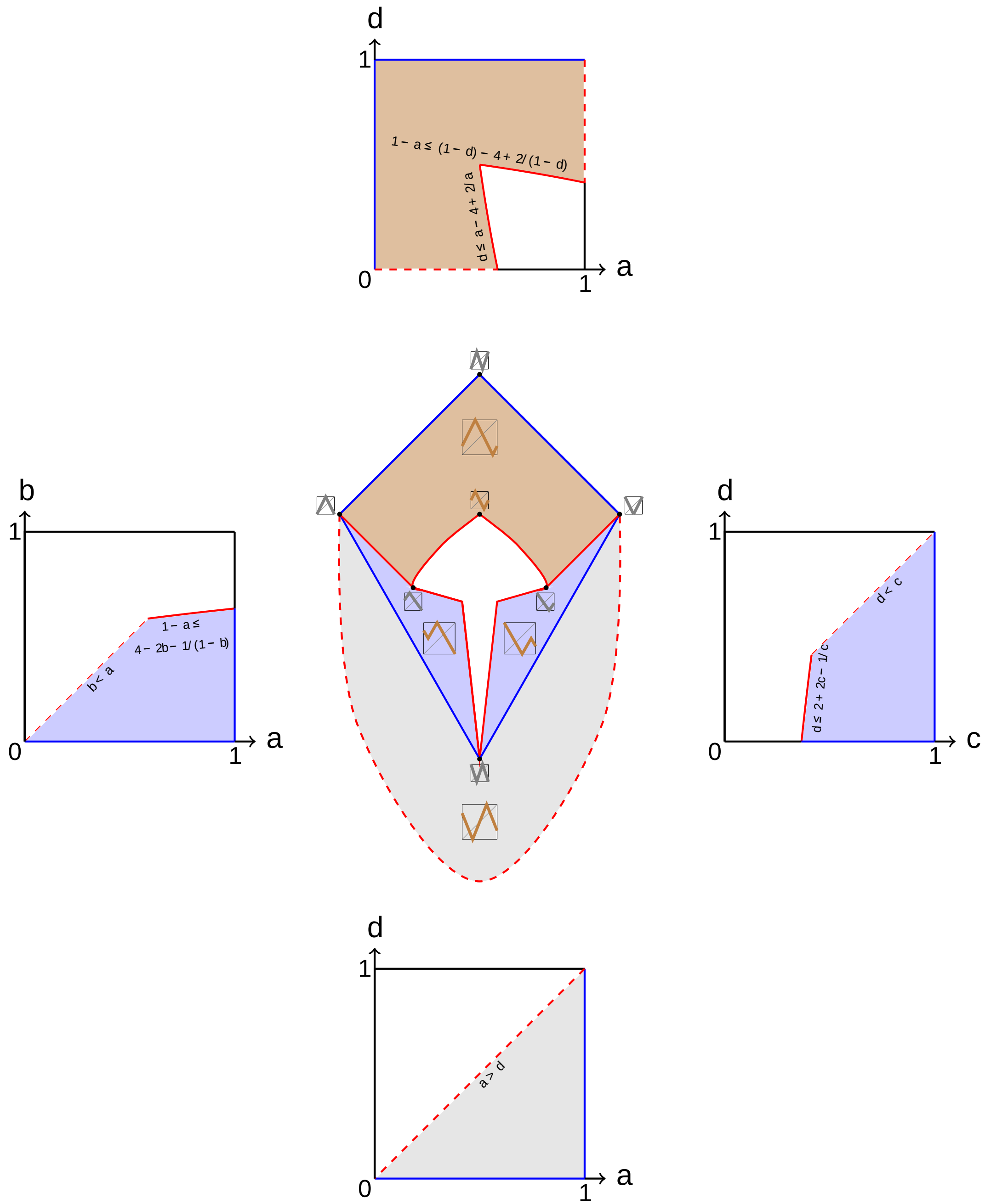
Lemma 8. *Let $f \in \mathcal{CS}_2$ have a code $(1, 0, c, d)$ where $c \in (0, 1]$ and $d \in [0, c)$. Then f is transitive if and only if*

$$d \leq 2 + 2c - \frac{1}{c} \ . \tag{2}$$

In the case $(a, b, 1, 0)$, if the fixed point in the third lap is x, then transitivity is equivalent to $b \leq x$:

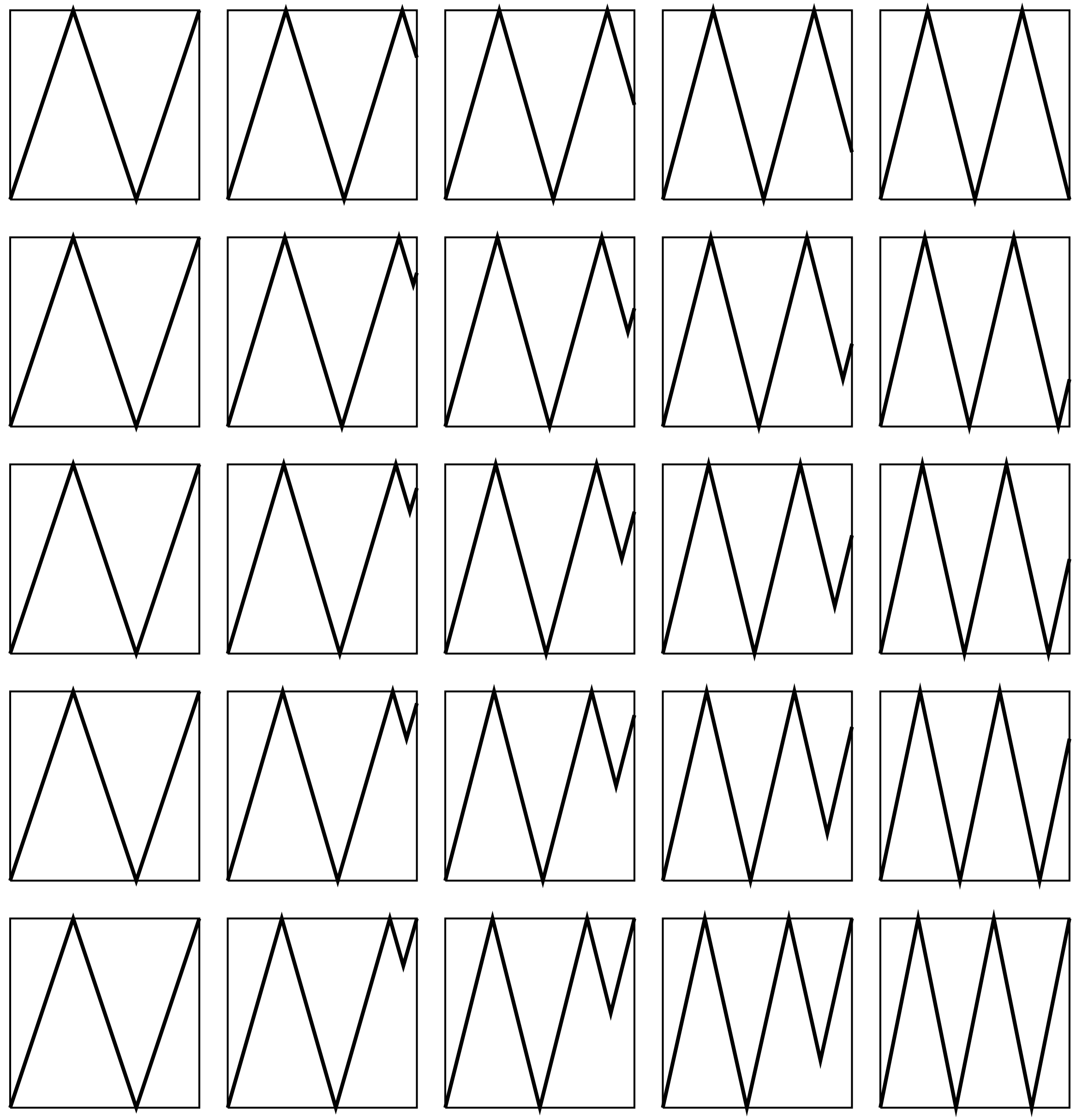
Lemma 9. *Let $f \in \mathcal{CS}_2$ have a code $(a, b, 1, 0)$ where $a \in (0, 1]$ and $b \in [0, a)$. Then f is transitive if and only if*

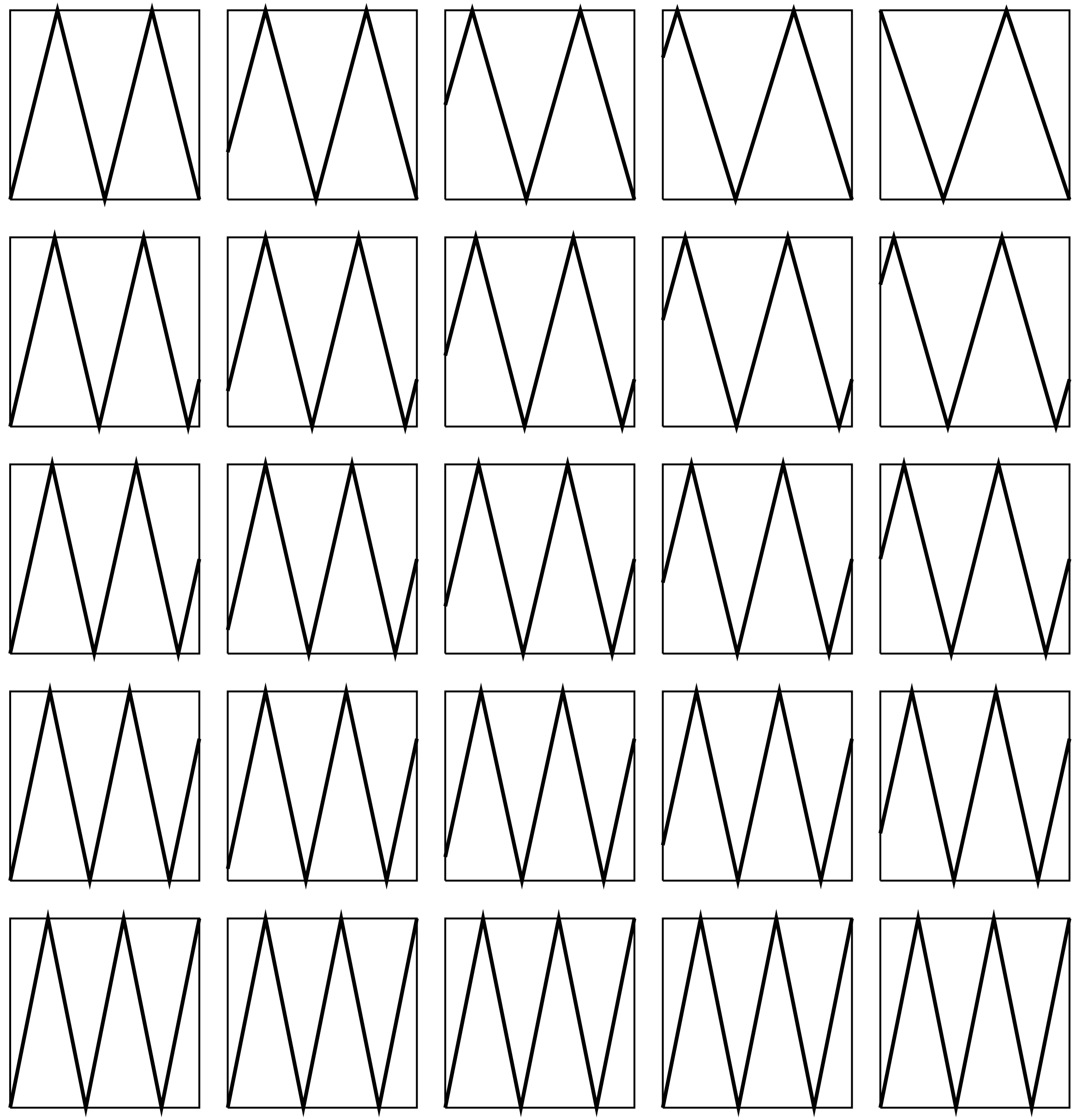
$$1 - a \leq 2 + 2(1 - b) - \frac{1}{1 - b} \ . \tag{3}$$

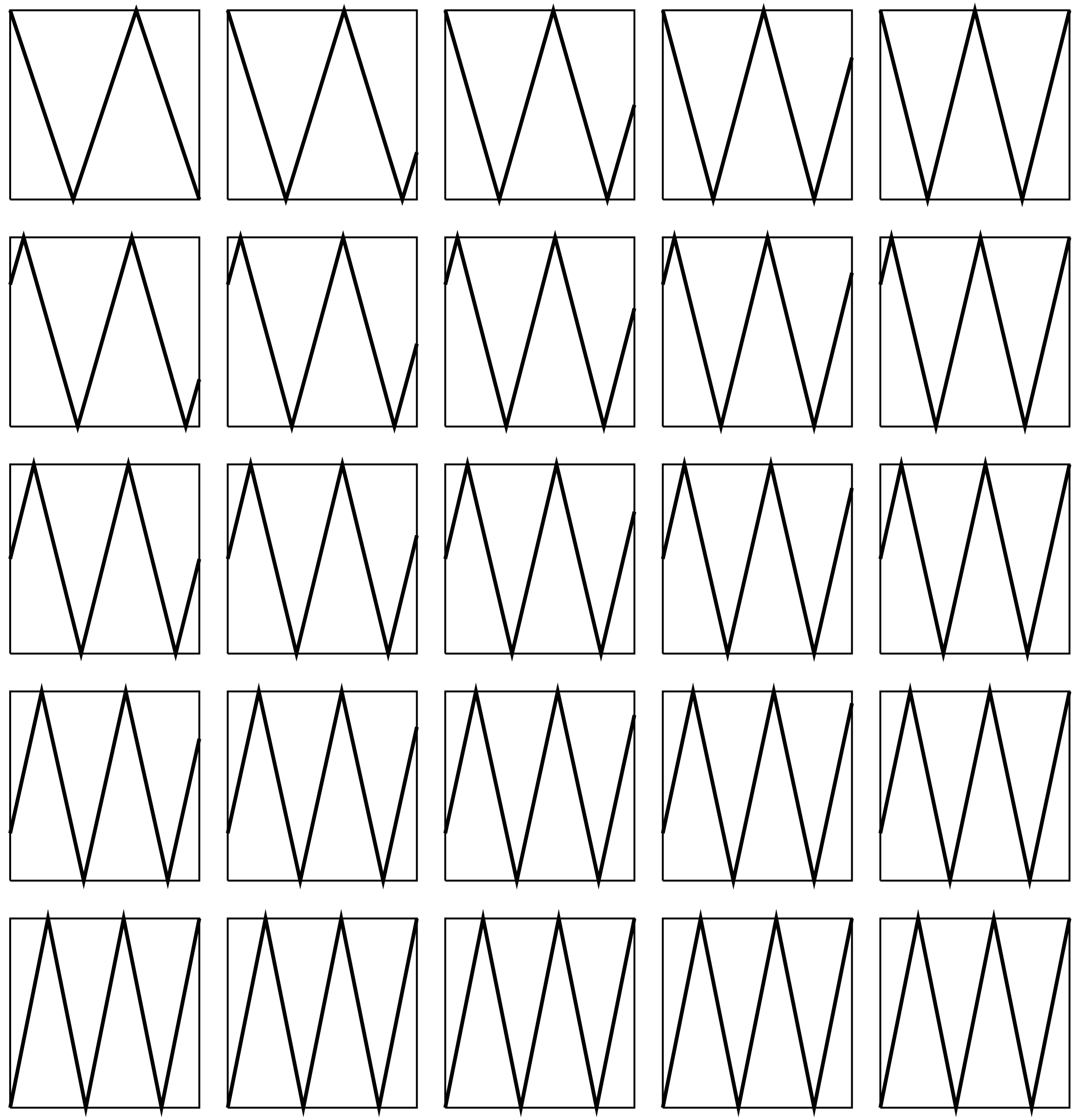

d
1
0
1
a
1 − a ≤ (1 − d) − 4 + 2/ (1 − d)
d ≤ a − 4 + 2/ a
b
1
0
1
a
1 − a ≤
4 − 2b − 1/ (1 − b)
b < a
d
1
0
1
c
d < c
d ≤ 2 + 2c − 1/ c
d
1
0
1
a
a > d

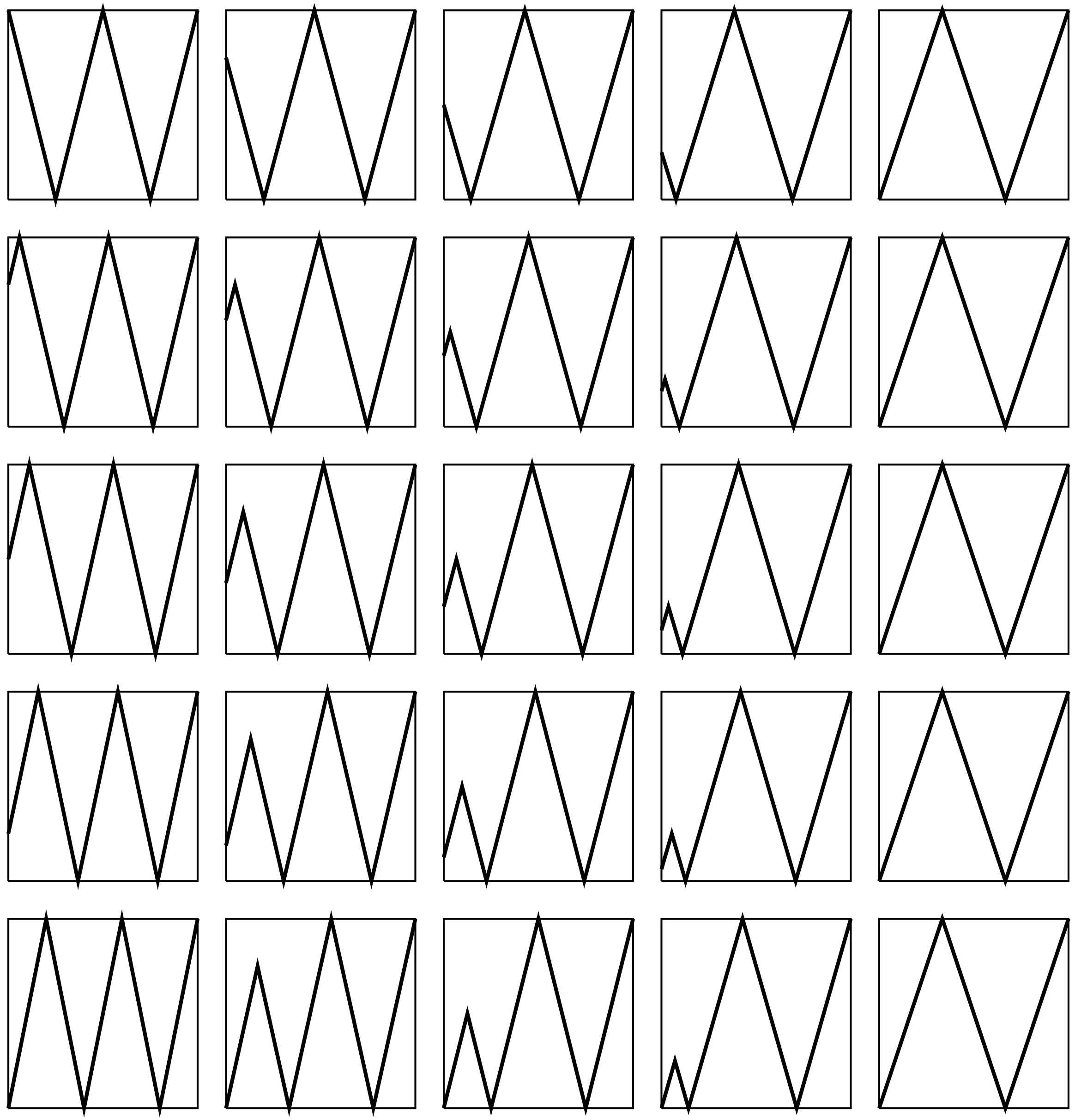
In order to contract the basic loop of order n in the space $\mathcal{TCS}_n \cup \mathcal{TCS}_{n+1} \cup \mathcal{TCS}_{n+2}$, we first deform it to the *auxiliary loop of order* n. I show how it works for $n = 2$. Since the basic loop consists of 4 arc, I will show 4 figures.

Observe that two of the four arcs of the auxiliary loop are constant, so we can cut them out.



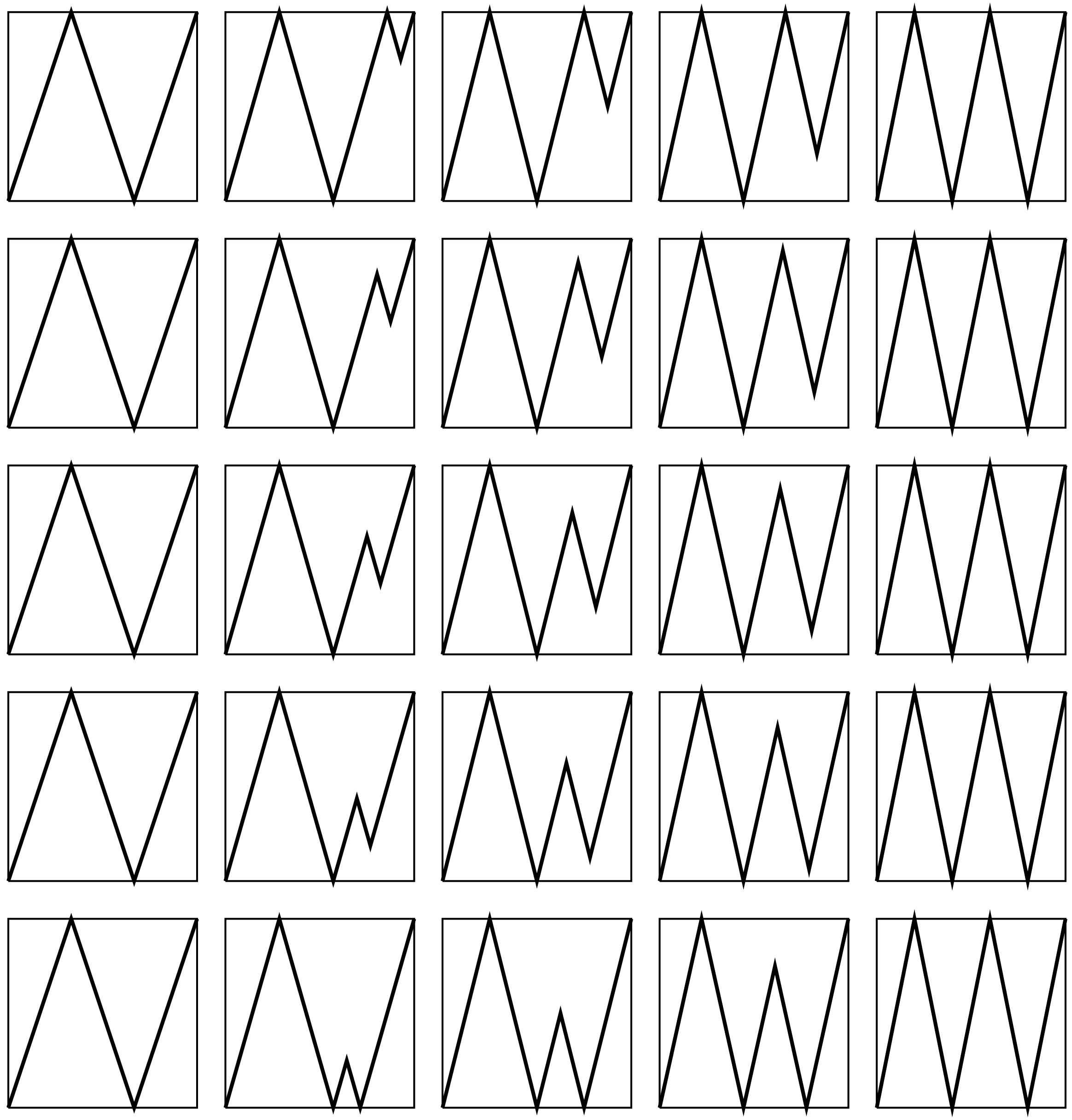






Using coding, we may describe the arcs of the loops as segments, parametrized by a variable s varying from 0 to 1. For instance, the first arc of the basic loop of order 2 is $(0, 1, 0, 1, 1 - s, 1 - s)$. For the deformation of this arc we use a parameter t, also varying form 0 (for the basic loop) to 1 (for the auxiliary loop); for instance $(0, 1, 0, 1, 1 - s, 1 - s + st)$.

The first arc of the auxiliary loop can be interpreted as a tooth consisting of the last two laps growing from nothing to full laps. This arc is homotopic to an arc where a tooth consisting of the next two laps (counting from the right) is growing. This is illustrated by the figure:



Now by the same techniques we homotop this arc to an arc where a tooth consisting of the next two laps is growing, etc. We continue this, and we get the arc where a tooth consisting of the two leftmost laps is growing. However, this is the second arc of the auxiliary loop run backward. This shows that the auxiliary loop is contractible.

Of course, all this requires proofs that the maps we are using are transitive. This follows from many lemmas. Let me cite three of them.

Lemma 10. *Let $f \in \mathcal{CS}_n$ have slope $\lambda > 2$, and assume that the image of every lap of f (except perhaps the leftmost and the rightmost ones) is the whole I. Then f is transitive.*

Lemma 11. *Let $f \in \mathcal{CS}_n$ have slope $\lambda > 3$, and assume that the image of every lap, except perhaps one or two leftmost or one or two rightmost ones, is the whole I. Then f is transitive.*

Denote by $E(f)$ the set consisting of all turning points of f and the endpoints of the interval.

Lemma 12. *Let $f \in \mathcal{CS}_n$ have slope $\lambda > 3$, and assume that $f(\{0,1\}) \subset \{0,1\}$. Assume also that out of any four consecutive points of $E(f)$ at least one is mapped to 0 and at least one to 1. Then f is transitive.*

We can contract the basic loop of order 2 in $\mathcal{TCS}_n \cup \mathcal{TCS}_{n+1} \cup \mathcal{TCS}_{n+2}$ in a different way. You can think about it as taking the mirror images of the previous figures. On the level of coding, we use different affine parametrizations of the arcs of the basic loop with different codings of the endpoints of those arcs.

Thus, the basic loop of order 2 is contained in a subset of $\mathcal{TCS}_n \cup \mathcal{TCS}_{n+1} \cup \mathcal{TCS}_{n+2}$ homeomorphic to the 2-dimensional sphere. It turns out that the basic loop of order 3 is also contained in this sphere, so it is also contractible in $\mathcal{TCS}_n \cup \mathcal{TCS}_{n+1} \cup \mathcal{TCS}_{n+2}$.

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