Periods of (continuous) maps and homeomorphisms on closed surfaces

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Tossa
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In what follows $\mathbb{M}_{g,0} = \mathbb{M}_g$.

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Basic definitions

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$\text{Per}(f)$ denotes the set of periods of all periodic points of f.

The aim of the present paper is to provide some information on $\text{Per}(f)$ using the Lefschetz fixed point theory.

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J.L.G. GUIRAO AND J. LLIBRE, Periods of continuous maps on closed surfaces, preprint, 2014.

J.L.G. GUIRAO AND J. LLIBRE, Periods of homeomorphisms on closed surfaces, to appear in "Discrete dynamical systems and applications", Proc. of ICDEA2012, Eds. L. Alseda, J. Cushing, S. Elaydi and A. Pinto, pp. 1–6.

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Then characteristic polynomial of A is

$$\det(tI - A) = t^n - E_1(A)t^{n-1} + E_2(A)t^{n-2} - \ldots + (-1)^n E_n(A).$$

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A and (d) are the integral matrices of the endomorphisms

$$f_{*i} : H_i(\mathbb{X}, \mathbb{Q}) \to H_i(\mathbb{X}, \mathbb{Q})$$

induced by f on the i–th homology group of $\mathbb{X}$, $i = 1, 2$.

SURFACE HOMEOMORPHISMS THEOREM

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If $\mathbb{X} = \mathbb{M}_{g,b}$ with $b > 0$, then

(c) If $E_1(A) = 1$, $E_2(A) = 0$, $2g + b - 1 \geq 3$ and k is the smallest integer of the set $\{3, 4, ..., 2g + b - 1\}$ such that $E_k(A) \neq 0$, then $\mathrm{Per}(f)$ has a periodic point of period a divisor of k.

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J. FRANKS AND J. LLIBRE, Periods of surface homeomorphisms, Contemporary Mathematics **117** (1991), 63–77.

Let $f : \mathbb{X} \to \mathbb{X}$ be a continuous map and let $\mathbb{X}$ be either $\mathbb{M}_{g,b}$ or $\mathbb{N}_{g,b}$.

Lefschetz fixed point theory

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The Lefschetz number of f is defined by

$$L(f) = \mathrm{trace}(f_{*0}) - \mathrm{trace}(f_{*1}) + \mathrm{trace}(f_{*2}).$$

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LEFSCHETZ FIXED POINT THEOREM
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The Lefschetz zeta function

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In order to study the whole sequence $\{L(f^n)\}_{n\geq 1}$ it is defined the formal Lefschetz zeta function of f as

$$Z_f(t) = \exp\left(\sum_{n=1}^{\infty} \frac{L(f^n)}{n} t^n\right).$$

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For a continuous self–map of a closed surface the Lefschetz zeta function is the rational function

$$Z_f(t) = \frac{\det(I - tf_{*1})}{\det(I - tf_{*0})\det(I - tf_{*2})}.$$

The Lefschetz zeta function for surface homeomorphisms I

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For an orientation–preserving homeomorphism $f : \mathbb{M}_{g,b} \to \mathbb{M}_{g,b}$ we have

$$Z_f(t) = \begin{cases} \dfrac{\det(I - tA)}{(1-t)^2} & \text{if } b = 0. \\ \dfrac{\det(I - tA)}{1-t} & \text{if } b > 0, \end{cases}$$

where $f_{*1} = A$ and $f_{*2} = (1)$ if $b = 0$, and $f_{*2} = (0)$ if $b > 0$.

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For an orientation–reversing homeomorphism $f : \mathbb{M}_{g,b} \to \mathbb{M}_{g,b}$ we have

$$Z_f(t) = \begin{cases} \dfrac{\det(I - tA)}{1-t^2} & \text{if } b = 0. \\ \dfrac{\det(I - tA)}{1-t} & \text{if } b > 0. \end{cases}$$

where $f_{*2} = (-1)$ if $b = 0$, and $f_{*2} = (0)$ if $b > 0$.

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For a homeomorphism $f : \mathbb{N}_{g,b} \to \mathbb{N}_{g,b}$ we have

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Note that for $b > 0$ for the surfaces $\mathbb{M}_{g,b}$ and for $b \geq 0$ for the surfaces $\mathbb{N}_{g,b}$ the expression of the zeta function is the same for all the previous three kind of homeomorphisms.

Proof of statement (a) of the SURFACE HOMEOMORPHISMS THEOREM I

$$
\begin{aligned}
&\sum_{n=1}^{\infty} \frac{L(f^n)}{n} t^n = \\
&= \log(Z_f(t)) \\
&= \log\left(\frac{\det(I - tA)}{1 - t}\right) \\
&= \log\left(\frac{1 - E_1(A)t + E_2(A)t^2 - \ldots + (-1)^m E_m(A)t^m}{1 - t}\right) \\
&= \log(1 - E_1(A)t + E_2(A)t^2 - \ldots) - \log(1 - t) \\
&= \left(-E_1(A)t + \left(E_2(A) - \frac{E_1(A)^2}{2}\right) t^2 - \ldots\right) - \left(-t - \frac{t^2}{2} - \ldots\right) \\
&= (1 - E_1(A))t + \left(\frac{1}{2} - \frac{E_1(A)^2}{2} + E_2(A)\right) t^2 + O(t^3).
\end{aligned}
$$

Proof of statement (a) of the SURFACE HOMEOMORPHISMS THEOREM II

Therefore we have

$$L(f) = 1 - E_1(A), \quad \text{and} \quad L(f^2) = 1 - E_1(A)^2 + 2E_2(A).$$

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(a) If $E_1(A) \neq 1$, then $1 \in \mathrm{Per}(f)$.

Hence, if $E_1(A) \neq 1$ then $L(f) \neq 0$, and by the Lefschetz fixed point theorem statement (a) follows.

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(b) If $E_1(A) = 1$ and $E_2(A) \neq 0$, then $\mathrm{Per}(f) \cap \{1, 2\} \neq \emptyset$.

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(b) If $E_1(A) = 1$ and $E_2(A) \neq 0$, then $\mathrm{Per}(f) \cap \{1, 2\} \neq \emptyset$.

If $E_1(A) = 1$ and $E_2(A) \neq 0$, then $L(f^2) = 2E_2(A) \neq 0$, and again by the Lefschetz fixed point theorem we get that $\mathrm{Per}(f) \cap \{1, 2\} \neq \emptyset$. So statement (b) is proved.

Proof of statement (c) of the SURFACE HOMEOMORPHISMS THEOREM I

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Assume now that $\mathbb{X} = \mathbb{M}_{g,b}$ with $b > 0$, $E_1(A) = 1$, $E_2(A) = 0$, $2g + b - 1 \geq 3$ and k is the smallest integer of the set $\{3, 4, ..., 2g + b - 1\}$ such that $E_k(A) \neq 0$.

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Therefore

$$\sum_{n=1}^{\infty} \frac{L(f^n)}{n} t^n =$$

$$= \log \left(\frac{1 - t + (-1)^k E_k(A) t^k + ... + (-1)^{b-1} E_{2g+b-1}(A) t^{2g+b-1}}{1 - t} \right)$$

$$= \log \left(1 + \frac{(-1)^k E_k(A) t^k + ... + (-1)^{b-1} E_{2g+b-1}(A) t^{2g+b-1}}{1 - t} \right)$$

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Hence $L(f) = ... = L(f^{k-1}) = 0$ and $L(f^k) = (-1)^k k E_k(A) \neq 0$.

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So, from the Lefschetz fixed point theorem, it follows the statement (c).

Proof of statement (d) of the SURFACE HOMEOMORPHISMS THEOREM I

Suppose that $\mathbb{X} = \mathbb{N}_{g,b}$ with $b \geq 0$, $g + b - 1 \geq 3$, $E_1(A) = 1$, $E_2(A) = 0$ and k is the smallest integer of the set $\{3, 4, ..., g + b - 1\}$ such that $E_k(A) \neq 0$. Therefore

$$\sum_{n=1}^{\infty} \frac{L(f^n)}{n} t^n =$$

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If $\mathbb{X}$ is either $\mathbb{M}_{g,b}$ with $b > 0$, or $\mathbb{N}_{g,b}$ with $b \geq 0$, then

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SURFACE MAPS THEOREM

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(c) If $E_1(A) \neq 1 + d$, then $1 \in \mathrm{Per}(f)$.

(d) If $E_1(A) = 1 + d$ and $E_2(A) \neq d^2 + d + 1$, then $\mathrm{Per}(f) \cap \{1, 2\} \neq \emptyset$.

If $\mathbb{X} = \mathbb{M}_{g,b}$ with $b > 0$, then

SURFACE MAPS THEOREM (continuation)

If $\mathbb{X} = \mathbb{M}_{g,b}$ with $b > 0$, then

(e) If $E_1(A) = 1$, $E_2(A) = 0$, $2g + b - 1 \geq 3$ and k is the smallest integer of the set $\{3, 4, ..., 2g + b - 1\}$ such that $E_k(A) \neq 0$, then Per(f) has a periodic point of period a divisor of k.

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(f) If $E_1(A) = 1$, $E_2(A) = 0$, $g + b - 1 \geq 3$ and k is the smallest integer of the set $\{3, 4, ..., g + b - 1\}$ such that $E_k(A) \neq 0$, then Per(f) has a periodic point of period a divisor of k.

THANK YOU VERY MUCH FOR YOUR ATTENTION