Entropy and independence in symbolic dynamics

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based on a joint work with
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An Enigma

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The Möbius function

Let $n \in \mathbb{N} = \{1, 2, \ldots \}$. For $n \geq 2$ let $\langle n \rangle$ be the number of distinct prime factors of $n$.

\[
\mu(n) = \begin{cases} 
1, & \text{if } n = 1, \\
0, & \text{if } n \text{ is not square-free}, \\
(-1)^{\langle n \rangle}, & \text{if } n \text{ is square-free}.
\end{cases}
\]

Is the sequence $(\mu(n))_{n=1}^{\infty}$ random?
Given a sequence \( x = (x_i)_{i=1}^{\infty} \in \{0, 1\}^\mathbb{N} \) we consider the shift space \( X \) which is the closure of the orbit of \( x \) with respect to the shift map \( \sigma \), that is
\[
X = \overline{\{\sigma^n(x) : n \geq 0\}}.
\]
From sequences to dynamical systems

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$$X = \overline{\{\sigma^n(x) : n \geq 0\}}.$$

**Definition**
A square-free flow is a shift space generated as above by the sequence $\eta(n) = \mu^2(n)$. 
Which sequences are random?

A heuristic is: a sequence is random if it is orthogonal to a deterministic one. Which sequences are deterministic? Those who come from a zero entropy homeomorphisms.
Shift spaces

Definition

Let $A = \{0, 1, \ldots, r - 1\}$ be an alphabet. The full $r$-shift is $A^\mathbb{N} = \{x = (x_i)_{i=1}^\infty : x_i \in A \text{ for all } i \in \mathbb{N}\}$.

The shift map $\sigma : A^\mathbb{N} \rightarrow A^\mathbb{N}$ maps $x = (x_i)_{i=1}^\infty$ to the sequence $\sigma(x) = (x_i+1)_{i=1}^\infty$.

A shift space is any closed $\sigma$-invariant subset of $A^\mathbb{N}$. 
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The shift map $\sigma : \mathcal{A}^\mathbb{N} \mapsto \mathcal{A}^\mathbb{N}$ maps $x = (x_i)_{i=1}^\infty$ to the sequence $\sigma(x) = (x_{i+1})_{i=1}^\infty$. A **shift space** is any closed $\sigma$-invariant subset of $\mathcal{A}^\mathbb{N}$. 
The language of a shift space

A block over $A$ is a finite sequence of symbols from $A$. An $n$-block stands for a block of length $n$.

The set of all blocks over $A$ is denoted by $A^*$. Let $x = (x_i)_{i=1}^{\infty} \in A^N$.

A block $w \in A^*$ occurs in $x$ if $w = x_{[i,j]}$ for some $1 \leq i < j < \infty$.

Definition: A language of a shift space $X$ is the set $Bl(X)$ of all blocks which do occur in some sequence $x \in X$. We write $B_n(X)$ for the set of all $n$-blocks contained in $B(X)$. 
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$$
\cdots x_{i-1} \quad \framebox{\begin{array}{c} x_i \ x_{i+1} \cdots x_{j-1} \ x_j \\ x[i,j] \end{array}} \ x_{j+1} \cdots
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$x[i,j]$\[x[i,j]\]

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Topological entropy
Definition
The entropy of a shift space $X$ is

$$h(X) = \lim_{n \to \infty} \frac{1}{n} \log |\mathcal{B}_n(X)| = \inf_{n \geq 1} \frac{1}{n} \log |\mathcal{B}_n(X)|.$$
Subordinate shift

Definition
A block \( w = w_1 \ldots w_k \in A^* \) dominates a block \( v = v_1 \ldots v_k \in A^* \) if \( v_i \leq w_i \) for \( i = 1, \ldots, k \) (\( y_i \leq x_i \) for \( i \in N \)).

Definition
A subordinate of \( L \subset A^* \) is the set \( L \leq \) of all \( v \in A^* \) that are dominated by some \( w \in L \).

Definition
Given a point \( x \in A^N \), a subordinate shift of \( x \), denoted by \( X \leq x \), is a shift space given by the language \( B \leq x \), where \( B_x \) is the language of \( x \). (Remark: Subordinate shifts are hereditary.)
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A subordinate of $L \subset \mathcal{A}^*$ is the set $L^\leq$ of all $v \in \mathcal{A}^*$ that are dominated by some $w \in L$.

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Given a point $x \in \mathcal{A}^\mathbb{N}$, a subordinate shift of $x$, denoted by $X^\leq x$, is a shift space given by the language $B^\leq_x$, where $B_x$ is the language of $x$. (Remark: Subordinate shifts are hereditary.)
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(Remark: Subordinate shifts are *hereditary*.)
Examples

Lemma

Square-free flow is a subordinate shift. As a consequence the combinatorial patterns appearing in its characteristic function are chaotic...
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Independence set for a shift space

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We say that a set $J \subset \mathbb{N}$ is an independence set for a shift space $X \subset A^\mathbb{N}$ if for every function $\phi : J \rightarrow A$ there is a point $x = \{x_j\}_{j=1}^\infty \in X$ such that $x_j = \phi(j)$ for every $j \in J$. 
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Examples of independence sets

Example
Any subset of \( \mathbb{N} \) is an independence set for the full shift.

Example
A subset of \( \mathbb{N} \) is an independence set for the golden mean shift if and only if it does not contain two consecutive integers.

Example
If \( J \subset \mathbb{N} \) is an independence set for a shift space \( X \), then so is every subset of \( J \).

Example
If \( X \) is a binary hereditary shift, then a \( J \subset \mathbb{N} \) is an independence set for \( X \) if and only if its characteristic function belongs to \( X \).
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Densities

Definition
A set $A \subset \mathbb{N}$ has density $\alpha$ if the limit $d(A) = \lim_{n \to \infty} \frac{|A \cap \{1, 2, \ldots, n\}|}{n}$ exists and is equal to $\alpha$.

Definition
The Shnirelman density of a set $A \subset \mathbb{N}$ is $d_{Sh}(A) = \inf_{n \geq 1} \left\{ \frac{|A \cap \{1, 2, \ldots, n\}|}{n} : n \in \mathbb{N} \right\}$. 
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Let $X$ be a binary shift. Then the entropy of $X$ is positive if and only if $X$ is independent over a set $A$ whose density exists, is positive, and is equal to its Shnirelman density.
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Limiting frequency

Let $a \in \mathbb{A}$ and $w=w_1...w_k \in \mathbb{A}^*$. Define $||w||_a = |\{1 \leq j \leq k : w_j = a\}|$, $M_{a^k}(X) = \max\{||w||_a : w \in B_{a^k}(X)\}$. The sequence $\{M_{a^k}(X)\}_{k=1}^{\infty}$ is subadditive, that is $0 \leq M_{a^m+n}(X) \leq M_{a^m}(X) + M_{a^n}(X)$, $m \in \mathbb{N}$. We define the limiting frequency of $a$ in $X$ by $\text{Fr}_a(X) = \lim_{k \to \infty} M_{a^k}(X)$.
Limiting frequency

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The sequence $\{M_k^a(X)\}_{k=1}^{\infty}$ is subadditive, that is

$$0 \leq M_{m+n}^a(X) \leq M_m^a(X) + M_n^a(X) \quad n, m \in \mathbb{N}.$$
**Limiting frequency**

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We define the limiting frequency of $a$ in $X$ by

$$\text{Fr}_a(X) = \lim_{k \to \infty} \frac{M^a_k(X)}{k} = \inf_{k \geq 1} \frac{M^a_k(X)}{k}.$$
A topological consequence of maximal ergodic theorem

Definition
For $a \in A$ and $x \in X$ define the characteristic set $\chi_a(x)$ as the set of positions at which $a$ appears in $x$, that is, $\chi_a(x) = \{ j \in \mathbb{N} : x_j = a \}$. 

Theorem
Let $X$ be a shift space over an alphabet $A$. Then for every symbol $a \in A$ there exists a point $\omega_a \in X$ such that $d_{Sh}(\chi_a(\omega_a)) = d(\chi_a(\omega_a)) = Fr_a(X)$. 
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Independence sets for blocks

Definition

Let $F$ be a (possibly empty) family of binary blocks of length $n \geq 0$. We say that $F$ is independent over a set $J \subset \mathbb{N}$ and $J$ is an independence set for $F$ if for each map $\phi: J \to \{0, 1\}$ there is a block $w \in F$ whose $i$-th symbol is $\phi(i)$ for every $i \in J$.

We denote the collection of all sets of independence for $F$ by $I(F)$. We assume the convention that the empty set is a set of independence for every (including empty) family of $n$-blocks.
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Sauer-Perles$^2$-Shelah Lemma

Let $F \subset \{0,1\}^n$ be a family of binary blocks of length $n \geq 1$. If for some $1 \leq k \leq n$ we have $|F| > k - 1 \sum_{j=0}^{\infty} \binom{n}{j}$, then $F$ is independent over some set of cardinality $k$.

Lemma (Pajor)

Let $F$ be a family of binary blocks of length $n \geq 0$. Then $|I(F)| \geq |F|$. 
Lemma (Sauer-Perles$^2$-Shelah)

Let $\mathcal{F} \subset \{0, 1\}^n$ be a family of binary blocks of length $n \geq 1$. If for some $1 \leq k \leq n$ we have

$$|\mathcal{F}| > \sum_{j=0}^{k-1} \binom{n}{j},$$

then $\mathcal{F}$ is independent over some set of cardinality $k$. 
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Lemma (Pajor)

Let $\mathcal{F}$ be a family of binary blocks of length $n \geq 0$. Then $|\mathcal{I}(\mathcal{F})| \geq |\mathcal{F}|$. 
Lemma (Calculus)

Let $0 < \varepsilon \leq 1/2$ and $n \geq 1$. Then

$$\sum_{j=0}^{\lfloor n\varepsilon \rfloor} \binom{n}{j} \leq 2^{n\cdot H(\varepsilon)},$$

where $H(\varepsilon) = -\varepsilon \log \varepsilon - (1 - \varepsilon) \log(1 - \varepsilon)$. 
Karpovsky-Milman

**Lemma (Karpovsky-Milman)**

Let $X$ be a binary shift with positive topological entropy. Then there is an $\varepsilon > 0$ such that for every $n \geq 1$ there is a set $J \subset \{1, \ldots, n\}$ with $\lfloor \varepsilon n \rfloor$ elements which is an independence set for $X$.

**Lemma (Calculus)**

Let $0 < \varepsilon \leq 1/2$ and $n \geq 1$. Then

$$\sum_{j=0}^{\lfloor n\varepsilon \rfloor} \binom{n}{j} \leq 2^{n \cdot H(\varepsilon)},$$

where $H(\varepsilon) = -\varepsilon \log \varepsilon - (1 - \varepsilon) \log(1 - \varepsilon)$. 
Theorem

If $X \preceq x$ is a subordinate shift of $x$, then

$$\text{Fr}_1(x) \leq h(X \preceq x) \leq \text{Fr}_1(x) + h(x).$$

In particular, if $h(x) = 0$, then $h(X \preceq x) = \text{Fr}_1(x)$.

Theorem

For every $t \in [0, 1]$ there is a binary subordinate shift with entropy $t$. 
Let $\mathcal{M}^\text{max}(X)$ be the set of all measures of maximal entropy for $X$.

**Theorem**

If $X \leq x$ is a subordinate shift of $x$ and $h(x) = 0$ then 
$\mathcal{M}^\text{max}(X) = \{\mu : \mu[1] = \text{Fr}_1(x)\}$. In particular, if $x$ is uniquely ergodic, then $X \leq x$ is intrinsically ergodic.

**Theorem**

There is a mixing binary subordinate shift with uncountably many measures of maximal entropy.