Entropy and independence in symbolic dynamics

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based on a joint work with
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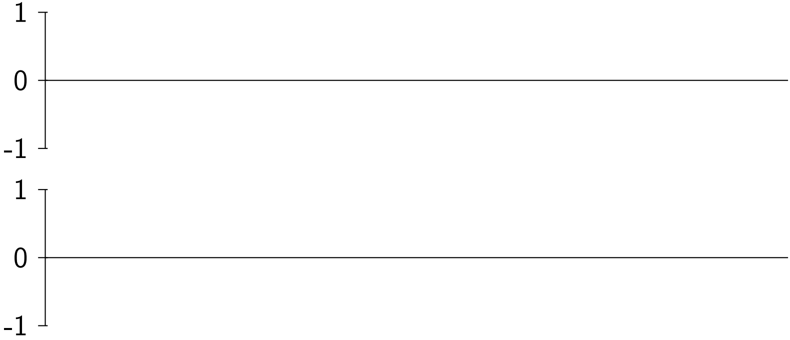
An Enigma



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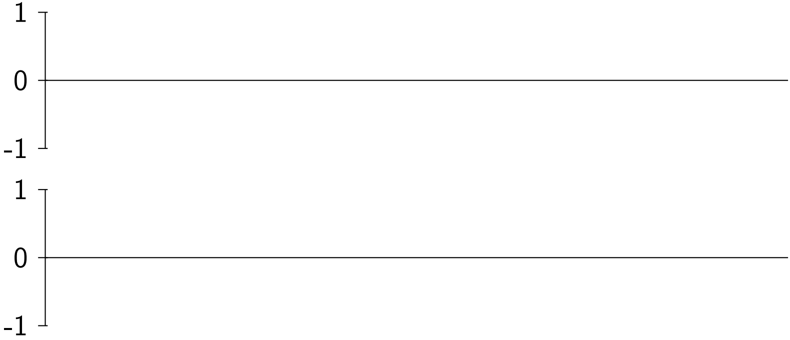

Which of these strings is generated at random?

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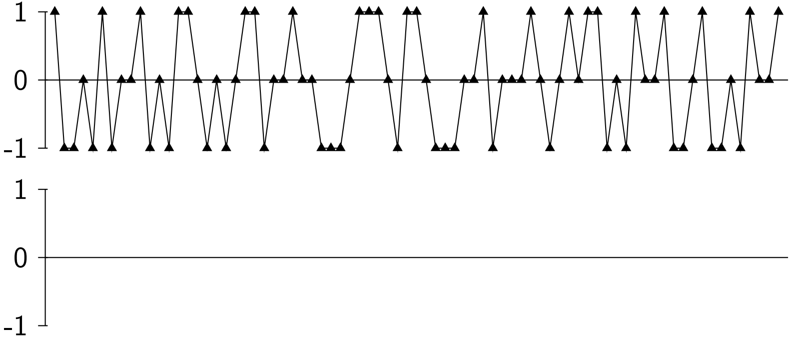

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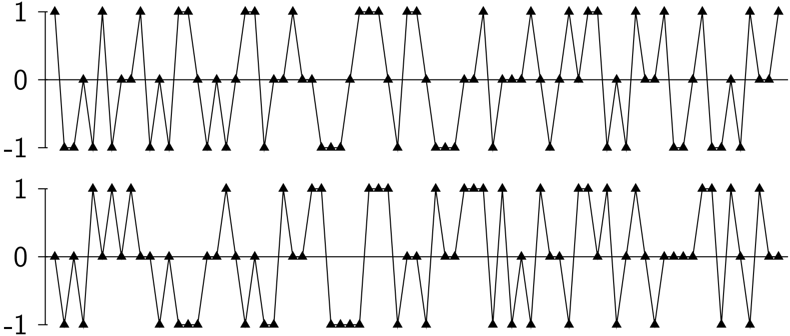

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The Möbius function

Let $n \in \mathbb{N} = \{1, 2, \ldots\}$. For $n \geq 2$ let $\langle n \rangle$ be the number of distinct prime factors of n.

$$\mu(n) = \begin{cases} 1, & \text{if } n = 1, \\ 0, & \text{if } n \text{ is not square-free}, \\ (-1)^{\langle n \rangle}, & \text{if } n \text{ is square-free}. \end{cases}$$

Is the sequence $(\mu(n))_{n=1}^{\infty}$ random?

From sequences to dynamical systems

Given a sequence $x = (x_i)_{i=1}^{\infty} \in \{0,1\}^{\mathbb{N}}$ we consider the shift space X which is the closure of the orbit of x with respect to the shift map σ, that is

$$X = \overline{\{\sigma^n(x) : n \geq 0\}}.$$

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Definition

A square-free flow is a shift space generated as above by the sequence $\eta(n) = \mu^2(n)$.

Which sequences are random?

A heuristic is: a sequence is random if it is orthogonal to a deterministic one. Which sequences are deterministic? Those who come from a zero entropy homeomorphisms.

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The shift map $\sigma \colon \mathcal{A}^{\mathbb{N}} \mapsto \mathcal{A}^{\mathbb{N}}$ maps $x = (x_i)_{i=1}^{\infty}$ to the sequence $\sigma(x) = (x_{i+1})_{i=1}^{\infty}$.

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The shift map $\sigma \colon \mathcal{A}^{\mathbb{N}} \mapsto \mathcal{A}^{\mathbb{N}}$ maps $x = (x_i)_{i=1}^{\infty}$ to the sequence $\sigma(x) = (x_{i+1})_{i=1}^{\infty}$. A shift space is any closed σ-invariant subset of $\mathcal{A}^{\mathbb{N}}$.

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A language of a shift space X is the set $Bl(X)$ of all blocks which do occur in some sequence $x \in X$. We write $\mathcal{B}_n(X)$ for the set of all n-blocks contained in $\mathcal{B}(X)$.

Topological entropy

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Definition

The entropy of a shift space X is

$$h(X) = \lim_{n\to\infty} \frac{1}{n} \log |\mathcal{B}_n(X)| = \inf_{n\geq 1} \frac{1}{n} \log |\mathcal{B}_n(X)|.$$

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A block $w = w_1 \dots w_k \in \mathcal{A}^*$ ($x \in \mathcal{A}^{\mathbb{N}}$) dominates a block $v = v_1 \dots v_k \in \mathcal{A}^*$ ($y \in \mathcal{A}^{\mathbb{N}}$) if $v_i \leq w_i$ for $i = 1, \dots, k$ ($y_i \leq x_i$ for $i \in \mathbb{N}$).

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Given a point $x \in \mathcal{A}^{\mathbb{N}}$, a subordinate shift of x, denoted by $X^{\leq x}$, is a shift space given by the language $\mathcal{B}_x^{\leq}$, where $\mathcal{B}_x$ is the language of x.

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(Remark: Subordinate shifts are *hereditary*.)

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As a consequence the combinatorial patterns appearing in its characteristic function are chaotic...

Independence set for a shift space

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Definition

We say that a set $J \subset \mathbb{N}$ is an independence set for a shift space $X \subset \mathcal{A}^{\mathbb{N}}$ if for every function $\varphi\colon J \to \mathcal{A}$ there is a point $x = \{x_j\}_{j=1}^{\infty} \in X$ such that $x_j = \varphi(j)$ for every $j \in J$.

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Example

If $J \subset \mathbb{N}$ is an independence set for a shift space X, then so is every subset of J.

Example

If X is a binary hereditary shift, then a $J \subset \mathbb{N}$ is an independence set for X if and only if its characteristic function belongs to X.

Densities

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Definition

A set $A \subset \mathbb{N}$ has density α if the limit

$$d(A) = \lim_{n \to \infty} \frac{|A \cap \{1, 2, \ldots, n\}|}{n}$$

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Definition

The Shnirelman density of a set $A \subset \mathbb{N}$ is

$$d_{Sh}(A) = \inf_{n \geq 1} \left\{ \frac{|A \cap \{1, 2, \dots, n\}|}{n} : n \in \mathbb{N} \right\}.$$

Main theorem

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Theorem

Let X be a binary shift. Then the entropy of X is positive if and only if X is independent over a set A whose density exists, is positive, and is equal to its Shnirelman density.

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The sequence $\{\mathsf{M}_k^a(X)\}_{k=1}^{\infty}$ is subadditive, that is

$$0 \leq \mathsf{M}_{m+n}^a(X) \leq \mathsf{M}_m^a(X) + \mathsf{M}_n^a(X) \quad n, m \in \mathbb{N}.$$

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We define the limiting frequency of a in X by

$$\mathrm{Fr}_a(X) = \lim_{k\to\infty} \frac{\mathrm{M}_k^a(X)}{k} = \inf_{k\geq 1} \frac{\mathrm{M}_k^a(X)}{k}.$$

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Definition

For $a \in \mathcal{A}$ and $x \in X$ define the characteristic set $\chi_a(x)$ as the set of positions at which a appears in x, that is,

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Theorem

Let X be a shift space over an alphabet $\mathcal{A}$. Then for every symbol $a \in \mathcal{A}$ there exists a point $\omega_a \in X$ such that

$$d_{Sh}(\chi_a(\omega_a)) = d(\chi_a(\omega_a)) = \mathrm{Fr}_a(X).$$

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Let $\mathcal{F}$ be a (possibly empty) family of binary blocks of length $n \geq 0$. We say that $\mathcal{F}$ is independent over a set $J \subset \mathbb{N}$ and J is an independence set for $\mathcal{F}$ if for each map $\varphi \colon J \to \{0, 1\}$ there is a block $w \in \mathcal{F}$ whose i-th symbol is $\varphi(i)$ for every $i \in J$. We denote the collection of all sets of independence for $\mathcal{F}$ by $\mathcal{I}(\mathcal{F})$. We assume the convention that the empty set is a set of independence for every (including empty) family of n-blocks.

Sauer-Perles^2-Shelah Lemma

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Lemma (Sauer-Perles^2-Shelah)

Let $\mathcal{F} \subset \{0,1\}^n$ be a family of binary blocks of length $n \geq 1$. If for some $1 \leq k \leq n$ we have

$$|\mathcal{F}| > \sum_{j=0}^{k-1} \binom{n}{j},$$

then $\mathcal{F}$ is independent over some set of cardinality k.

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Lemma (Pajor)

Let $\mathcal{F}$ be a family of binary blocks of length $n \geq 0$. Then $|\mathcal{I}(\mathcal{F})| \geq |\mathcal{F}|$.

Karpovsky-Milman

Lemma (Calculus)

Let $0 < \varepsilon \leq 1/2$ *and* $n \geq 1$. *Then*

$$\sum_{j=0}^{\lfloor n\varepsilon \rfloor} \binom{n}{j} \leq 2^{n \cdot H(\varepsilon)},$$

where $H(\varepsilon) = -\varepsilon \log \varepsilon - (1-\varepsilon)\log(1-\varepsilon)$.

Karpovsky-Milman

Lemma (Karpovsky-Milman)

Let X be a binary shift with positive topological entropy. Then there is an $\varepsilon > 0$ such that for every $n \geq 1$ there is a set $J \subset \{1, \ldots, n\}$ with $\lfloor \varepsilon n \rfloor$ elements which is an independence set for X.

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Entropy of subordinate shifts

Theorem

If $X^{\leq x}$ is a subordinate shift of x, then

$$\mathrm{Fr}_1(x) \leq h(X^{\leq x}) \leq \mathrm{Fr}_1(x) + h(x).$$

In particular, if $h(x) = 0$, then $h(X^{\leq x}) = \mathrm{Fr}_1(x)$.

Theorem

For every $t \in [0,1]$ there is a binary subordinate shift with entropy t.

Intrinsic ergodicity of subordinate shifts

Let $\mathcal{M}^{\mathsf{max}}(X)$ be the set of all measures of maximal entropy for X.

Theorem

If $X^{\leq x}$ is a subordinate shift of x and $h(x) = 0$ then $\mathcal{M}^{\mathsf{max}}(X) = \{\mu : \mu[1] = \mathsf{Fr}_1(x)\}$. In particular, if x is uniquely ergodic, then $X^{\leq x}$ is intrinsically ergodic.

Theorem

There is a mixing binary subordinate shift with uncountably many measures of maximal entropy.