# Metric entropy and stochastic laws of invariant measures for elliptic functions

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# Plan of the talk

1. Invariant measures for transcendental meromorphic functions
2. Basic properties of 'critically tame' elliptic functions
3. The results
4. Idea of the proofs (the main ingredients of the proofs)
   - thermodynamic formalism for conformal graph directed Markov systems
   - nice sets for holomorphic maps of the Riemann surfaces
   - L.S. Young's tower
   - stochastic properties of the return map
   - metric entropy

# Basic definition

## Fatou set and Julia set

Let $f : \mathbb{C} \to \hat{\mathbb{C}}$ be a transcendental meromorphic function or let $f : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ be a rational map of degree $\geq 2$. Then

- The Fatou set $F(f)$ is defined as usual using normal families: a point $z$ is in the Fatou set if and only in there is a neighbourhood of $z$ on which the iterates of $f$ are well defined and form a normal family.
- The Julia set $J(f) := \hat{\mathbb{C}} \setminus F(f)$.

## Basic definition

Let $S(f) \subset \hat{\mathbb{C}}$ denote the set of singular values of $f$:

- a point $z$ is an element of $\hat{\mathbb{C}} \setminus S(f)$ if and only if there is a neighbourhood of $z$ on which all inverse branches are well - defined, univalent maps. This set contains critical values and asymptotic values.

# Basic definition

## Definition

A point $a \in \hat{\mathbb{C}}$ is called an asymptotic value if there is a path $\gamma : [0,1) \to \mathbb{C}$ such that:

- $\lim_{t \to 1^-} \gamma(t) = \infty$ and
- $\lim_{t \to 1^-} f(\gamma(t)) = a$

## Examples

1. $f(z) = e^z \neq 0, \quad a_1 = 0 = \lim_{x \to -\infty} e^x$
2. $f(z) = \tan(z) \neq \pm i, \quad a_1 = i = \lim_{y \to +\infty} \tan z, \quad a_2 = -i = \lim_{y \to -\infty} \tan z$

## Definition

$\mathcal{P}(f) := \overline{\bigcup_{n \geq 0} f^n(S(f))} -$ is called the post - singular set.

### Theorem A (Urbański and Kotus)

Let $f : \mathbb{C} \to \hat{\mathbb{C}}$ be a transcendental meromorphic function satisfying the following conditions:

- $J(f) = \hat{\mathbb{C}}$
- $m(\{z \in J(f) \colon \omega(z)$ is not contained in $\mathcal{P}(f)\}) > 0$ (i.e. $\mathcal{P}(f)$ is not a metric attractor)

then there exists a $\sigma$-finite ergodic and conservative $f$ invariant measure $\mu$ equivalent with the Lebesgue measure $m$.

### Remark

We applied M. Martens' technique of construction of invariant measures

### Examples

(a) $f(z) = 2\pi i e^z$ (b) $f(z) = \pi i tan(z)$

# I - Probabilistic invariant measures

**Theorem B (Świątek and Kotus)**

1. Let $f : \mathbb{C} \to \overline{\mathbb{C}}$ be a meromorphic function with finitely many singular values. Suppose further that all poles of f have multiplicities bounded by M.
2. Suppose also that $J(f) = \overline{C}$ and $\mathcal{P}(f) \cap (Crit(f) \cup \{\infty\}) = \emptyset$.
3. for some $r_0 > 0$
$$\int_{r_0}^{\infty} \frac{m(r,a)}{r^{1+\frac{2}{M}}} dr < \infty,$$
for each asymptotic value $a$.

Then, f has a probabilistic ergodic and conservative invariant measure which is absolutely continuous with respect to the Lebesgue measure.

where $m(r,a) = \int_{r_0}^{2\pi} \log^{+} \frac{1}{\mathrm{dist}(f(re^{it}),a)} dr$- proximity function

# I - Probabilistic invariant measures

## Remarks

Theorem B does not apply to:

- entire functions
- meromomorphic funtions with finitely many poles

## Examples

Hypotheses of Theorem B are satisfied by the functions

- $f(z) = k\pi i \tan(z)$
  (Theorem A $\Rightarrow$ invariant measure is $\sigma$-finite)
- elliptic functions if $J(f) = \hat{\mathbb{C}}$ and $\mathcal{P}(f) \cap (\mathrm{Crit}(f) \cup \{\infty\}) = \emptyset$.

## Remark

The **necessity** of the hypothesis (3) $\int_{r_0}^{\infty} \frac{m(r,a)}{r^{1+\frac{2}{M}}} dr < \infty$

Let $f(z) = A\frac{e^{(z-a)^2}-1}{e^{(z-a)^2}+1}$. Then $f$ does not satisfy (3) and $f$ doesn't have finite invariant measure.

# I - Probabilistic invariant measures

## Theorem - Dobbs (a converse to a theorem of Świątek and Kotus)

1. Let $f : \mathbb{C} \to \overline{\mathbb{C}}$ be a meromorphic function such there exists a positive Lebesgue measure set of points $z \in J(f)$ such that $\omega(z)$ is not contained in $\mathcal{P}(f)$.
2. Let $A$ be a forward invariant, bounded set and suppose $f$ admits a pole of order $M$ which is not an omitted value.
3. If the $\sigma$ invariant measure given in Theorem A is finite, then

$$\int_{|z|>r_0} \frac{\operatorname{dist}(f(z), A)}{|z|^{2+\frac{2}{M}}} dm < \infty$$

for some $r_0$, where integration is with respect to Euclidean Lebesgue measure m.

## Remark

Suppose $f$ admits an asymptotic value a whose orbit is bounded. Let $A := Orb^+(a)$. Then

$$\int_{|z|>r_0} \frac{\mathrm{dist}(f(z), A)}{|z|^{2+\frac{2}{M}}} dm < \infty$$

implies

$$\int_{|z|>r_0} \frac{\mathrm{dist}(f(z), a)}{|z|^{2+\frac{2}{M}}} dm < \infty.$$

One can rewrite the inequality as

$$\int_{r>r_0}^{\infty} \frac{m(r,a)}{|r|^{1+\frac{2}{M}}} dr < \infty.$$

where $m(r,a) = \int_{r_0}^{2\pi} \log^+ \frac{1}{\mathrm{dist}(f(re^{it}),a)} dr$

# II- Elliptic functions

## Definition

Let $f : \mathbb{C} \to \overline{\mathbb{C}}$ be a non-constant elliptic function. Every such function is doubly periodic and meromorphic i.e. there exist two vectors $w_1$, $w_2$, $\mathrm{Im}(\frac{w_1}{w_2}) \neq 0$, such that for every $z \in \mathbb{C}$ and $n, m \in \mathbb{Z}$, $f(z) = f(z + mw_1 + nw_2)$.

## Remark

Every elliptic function has a form $R(\wp, \wp')$ where $R$ is rational, $\wp$ is a Weiestrass function.

$$\wp(z) = \frac{1}{z^2} + \frac{1}{z^2} + \sum_{k=1}^{\infty} \left[ \frac{1}{(z - w_k)^2} - \frac{1}{w_k^2} \right], \quad w_k = mw_1 + nw_2$$

## Definition

Let $f : \mathbb{C} \to \hat{\mathbb{C}}$ be an elliptic function and $c \in \mathrm{Crit}(f)$. We say that $f$ is critically tame if the following conditions are satisfied:

- if $c \in F(f)$- Fatou set, then there exists an attracting or parabolic cycle of period $p$, $O(z_0) = \{z_0, f(z_0), ..., f^{p-1}(z_0)\}$ such that $\omega$ limit set $\omega(c) = O(z_0)$.
- if $c \in J(f)$ - Julia set, then one of the following holds:
  - $\omega(c)$ is a compact subset of $\mathbb{C}$ such that $c \notin \omega(c)$; (i.e. non-recurrent property) but $c \in \omega(c')$ where $c' \in \mathrm{Crit}(f)$)
  - $c$ is eventually mapped onto some pole;
  - $\lim_{n\to\infty} f^n(c) = \infty$

# III - The results

**Theorem C - Urbański and Kotus**

Let $f$ be a non-constant elliptic function. Then

$$\dim_H(J(f)) > \frac{2q}{q+1} \geq 1$$

where $q$ is the maximal multiplicity of poles of $f$.

**Theorem D -Urbański and Kotus**

Let $f : \mathbb{C} \to \overline{\mathbb{C}}$ be a critically tame elliptic function.

- If $h = \dim_H(J(f)) = 2$, then $J(f) = \overline{\mathbb{C}}$.
- If $h < 2$, then
  1. $h$- dimensional Hausdorff measure $\mathrm{H}^h_s(J(f)) = 0$.
  2. $h$ -dimensional packing measure $\Pi^h_s(J(f)) > 0$.
  3. $\Pi^h_s(J(f)) = \infty$ if and only if $\Omega(f) \neq \emptyset$, $\Omega(f)$ is the set of parabolic periodic points.

**Theorem E- Urbański and Kotus**

Suppose that $f$ is critically tame elliptic function, denote $h = \dim_H(J(f))$. Then there exist:

- a unique atomless $h$-conformal measure $m$ for $f : J(f) \setminus \{\infty\} \to J(f)$ where $m$ is ergodic and $m(Tr(f)) = 1$; $Tr(f) \subset J(f)$ denotes the set of all transitive points of $f$
- if $f$ has no parabolic periodic points, then $0 < \Pi_s^h(J(f)) < \infty$ and $m$ and $\Pi_s^h$ are equivalent.
- there exists a non-atomic, $\sigma$-finite, ergodic and invariant measure $\mu$ for $f$, equivalent to the measure $m$. Additionally, $\mu$ is unique up to a multiplicative constant and is supported on $J(f)$.
- the Jacobian $D_\mu f = \frac{d\mu \circ f}{d\mu}$ has a real analytic extension on a neighborhood of $J(f) \setminus (\overline{\mathrm{PC}(f)} \cup f^{-1}(\infty))$ in $\mathbb{C}$.

## Conformal measure

Fix $t \geq 0$. Let $G$ and $H$ be non-empty open subsets of $\overline{\mathbb{C}}$. Let $f : G \to H$ be a meromorphic map.

A pair $(m_G, m_H)$ of Borel finite measures on $G$ and $H$ respectively is called spherical $t$-conformal pair of measures for the map $f : G \to H$, if

$$m_H(f(A)) = \int_A |f^*|^t \, dm_G$$

for every Borel set $A \subset G$ such that $f|_A$ is injective.

If both measures $m_G$ and $m_H$ are restrictions of the same Borel finite measure $m$ defined defined on $G \cup H$, we refer to $m$ as t- conformal measure the map $f : G \to H$.

# III - The results

## Definitions

Let $f : \mathbb{C} \to \overline{\mathbb{C}}$ be critically tame elliptic functions

- If $f$ has no parabolic periodic points
- and $\mathrm{Crit}_\infty(f) = \emptyset$ (no critical points diverge to infinity)

then f is called of finite character.

## Proposition

If $f : \mathbb{C} \to \overline{\mathbb{C}}$ is a critically tame elliptic of finite character then

- $\mu_h$ finite.
- in particular if Julia set is equal to the entire complex plane $\mathbb{C}$, then there exists a unique Borel probability $f$-invariant measure $\mu$ equivalent to the planar Lebesgue measure on $\mathbb{C}$. (as before in Theorem B)

# III - The new results

**Theorem 1(a) - Decay of correlation** - Urbański and Kotus

If $f : \mathbb{C} \to \overline{\mathbb{C}}$ is an elliptic function of finite character and if $\mu$ is the probability $f$-invariant measure equivalent to the $h$-conformal measure $m$, then for the dynamical system $(f, \mu)$ the following holds.

Fix $\alpha \in (0, 1]$ and a bounded function $g : J(f) \to \mathbb{R}$ which is Hölder continuous with respect to the Euclidean metric on $J(f)$ with the exponent $\alpha$. Then for every bounded measurable function $\psi : J(f) \to \mathbb{R}$, we have that

$$\left| \int \psi \circ f^n \cdot g d\mu - \int g dmu \int \psi d\mu \right| = O(\theta^n)$$

for some $0 < \theta < 1$ depending on $\alpha$.

**Theorem 1(b) - The Central Limit Theorem** - Urbański and Kotus

The Central Limit Theorem holds for every Hölder continuous function $g : J(f) \to \mathbb{R}$ that is not cohomologous to a constant in $L^2(\mu)$, i.e. for which there is no square integrable function $\eta$ for which $g = \mathrm{const} + \eta \circ f - \eta$. Precisely this means that there exists $\sigma > 0$ such that

$$\frac{1}{\sqrt{n}} S_n g = \frac{1}{\sqrt{n}} \sum_{j=0}^{n-1} g \circ f^j \to N(0, \sigma)$$

in distribution, where $N(0, \sigma)$ is here the normal (Gaussian) distribution with 0 mean and variance $\sigma$. Equivalently for every $t \in \mathbb{R}$,

$$\mu\left(\{x \in X : \frac{1}{\sqrt{n}} S_n g(x) \leq t\}\right) \to \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{t} \exp\left(-u^2/2\sigma^2\right) du.$$

# III - The new results

**Theorem 1(c)-The Law of Iterated Logarithm-** Urbański and Kotus

The Law of Iterated Logarithm holds for every Hölder continuous function $g : J(f) \to \mathbb{R}$ that is not cohomologous to a constant in $L^2(\mu)$. This means that there exists a real positive constant $A_g$ such that such that $\mu_\phi$ almost everywhere

$$\limsup_{n\to\infty} \frac{S_n g - n\int g d\mu}{\sqrt{n \log\log n}} = A_g.$$

**Theorem 2** - Urbański and Kotus

If $f : \mathbb{C} \to \overline{\mathbb{C}}$ is a critically tame map of finite type, $\mu_h$ is the corresponding Borel probability f-invariant measure equivalent to the h-conformal measure $m$,then a metric entropy

$$h_{\mu_h}(f) < +\infty.$$

# The main ingredients of the proof of Theorem 1

A) Thermodynamic formalism for graph directed Markov system

B) Nice sets for analytic maps

C) Young's tower technique

D) Stochastic properties of the return map

**Part I - projection onto $\mathbb{T}$**

- Let $f : \mathbb{C} \to \overline{\mathbb{C}}$ be a critically tame elliptic function
- Let $\mathbb{T} = \mathbb{C}/\sim_f$ (the torus generated by the lattice $\Lambda$ of $f$).
- $B(f) = f^{-1}(\infty) \cup (\mathrm{Crit}(f) \cap J(f))$ is infinite
- $\Pi : \mathbb{C} \to \mathbb{T}$ be the canonical projection, $\hat{\mathbb{T}} := \Pi(\mathbb{C} \setminus f^{-1}(\infty))$

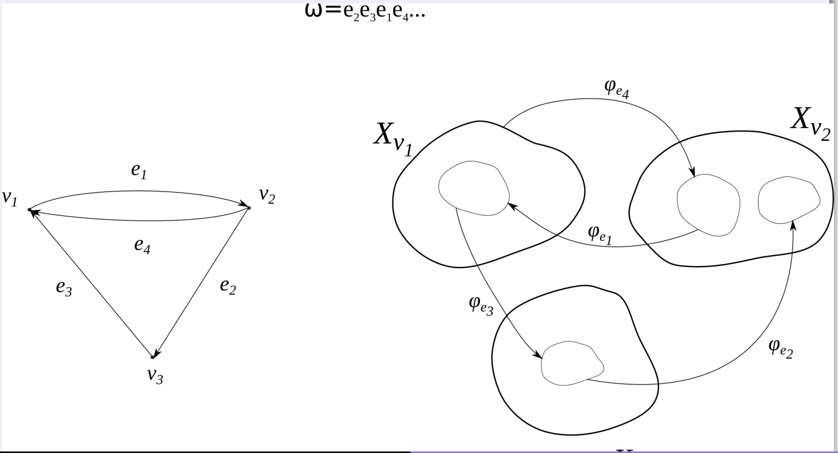
$$\begin{tikzcd}
\mathbb{C} \setminus f^{-1}(\infty) \arrow[r, "f"] \arrow[d, "\Pi"'] & \mathbb{C} \arrow[d, "\Pi"] \\
\hat{\mathbb{T}} \arrow[r, "\hat{f}"] & \mathbb{T}.
\end{tikzcd}$$

- Then $B(\hat{f}) = \Pi(B(f))$ is finite !!

# Idea of proof of stochastics laws

## Part I - projection onto $\mathbb{T}$

We construct Graph Directed Markow System such that its vertices are in $B(\hat{f}) = \Pi(B(f)) \in \mathbb{T}$



# Idea of proof of stochastics laws

## Part I - projection onto $\mathbb{T}$

A Graph Directed Markov System consists of

- a directed multigraph $(E, V)$ with a countable set of edges $E$ and a finite set of vertices $V$,
- an incidence matrix $A : E \times E \to \{0, 1\}$,
- two functions $i, t : E \to V$ such that $t(a) = i(b)$ whenever $A_{ab} = 1$.
- a family of non-empty compact metric spaces $\{X_v\}_{v \in V}$,
- a number $\beta \in (0, 1)$, and for every $e \in E$, a 1-to-1 contraction $\phi_e : X_{t(e)} \to X_{i(e)}$ with a Lipschitz constant $\leq \beta$.
- The set $\mathcal{S} = \{\phi_e : X_{t(e)} \to X_{i(e)}\}_{e \in E}$ is called a Graph Directed Markov System (GDMS).
- The set $J = J_S := \pi(E_A^\infty)$ is called the limit set of the Graph Directed Markov System $\mathcal{S} = \{\phi_e\}_{e \in E}$ (GDMS).

## Part I - projection onto $\mathbb{T}$

We prove:

- Bowen formula i.e. Hausdorff dimension of the limit set $J_S$ of Graph Directed Markov System $\mathcal{S}$ is equal to zero of the topological pressure.
- Graph Directed Markov System $\mathcal{S}$ corresponds to a subshift $(E_A^{\mathbb{N}}, \sigma)$ which has $h$- conformal measure $\tilde{m}_h$
- $m_h = \tilde{m}_h \circ \Pi^{-1}$ defines $h$- conformal measure on the limit set $J$ of Graph Directed Markov System.

# Idea of proof of stochastics laws

## Part II - 'lift' GDMS to $\mathbb{C}$

- For every $c \in (\mathrm{Crit}(f) \cap J(f))$, $f$ is elliptic function of finite type, we define Iterated Function System by 'lifting' some branches of GDMS defined on $\mathbb{T}$. So there is a limit set $J_c$ and a conformal measure $m_c$ defined on $J_c$.
- we show that $m_c$ is comparable with restriction of conformal measure $m$ to $J_c$, where $m$ was proved to exist for elliptic functions in Theorem E,
- we consider a return map $F$ on a neighbourhood $V$ of $c$ in the Julia set $J(f)$ and prove that the greatest common divisor of all return time numbers is equal to 1.

## Part II - 'lift' GDMS to $\mathbb{C}$

- we construct Young's tower associated with $V$ and return time map $F$
- we check that a return map satisfies the assumptions of L.S. Young theorems concerning stochastic laws of invariant measure, which implies the required properties of invariant measure for critically tame elliptic functions

# IV - metric entropy of critically tame elliptic functions

## Theorem 2

- If $f : \mathbb{C} \to \overline{\mathbb{C}}$ is a critically tame map of finite type,
- $\mu_h$ is the corresponding Borel probability f-invariant measure equivalent to the h-conformal measure $m$,
- then a metric entropy $h_{\mu_h}(f) < +\infty$.

## Corollary

- If $f : \mathbb{C} \to \overline{\mathbb{C}}$ is a critically tame elliptic function with $J(f) = \mathbb{C}$ and $\mathrm{Crit}_\infty(f) = \emptyset$,
- $\mu$ is the (unique) Borel probability $f$–invariant measure on $\mathbb{C}$ equivalent to the planar Lebesgue measure on $\mathbb{C}$ .
- then $\mathrm{h}_\mu(f) < +\infty$

## Theorem - Abramov

If $T : X \to X$ is an ergodic measure preserving transformation of a probability space $(X, \mathcal{F}, \mu)$, then for every set $K \in \mathcal{F}$ with $0 < \mu(K) < +\infty$, we have that

$$\mathrm{h}_{\mu_K}(F) = \frac{1}{\mu(K)} h_\mu(T).$$

where

- $F(x) := T^{\tau_K(x)}(x)$ is an induced map
- $\tau_K(x) := \min\{n \geq 1 : \ T^n(x) \in K\}$.
- $\mu_K := \mu_{|K}(\mu(K))^{-1}$

### Krengel's Entropy

If $T : X \to X$ is an conservative ergodic measure preserving transformation of a measure space $(X, \mathcal{F}, \mu)$, then for all sets $F$ and $G$ in $\mathcal{F}$ with $0 < \mu(F), \mu(G) < +\infty$, we have that $\mathrm{h}_{\mu_F}(T_F) = \mathrm{h}_{\mu_G}(T_G)$.

- This common value is called the Krengel' entropy of the map $T : X \to X$ and is denoted simply by $\mathrm{h}_\mu(T)$.
- If $\mu$ is a probability measure, it coincides with the standard entropy of $T$ with respect to $\mu$.

## The proof of Theorem 2

- Abramov's formula gives $h_{\mu_F}(F) = \frac{1}{\mu(J_c)} h_\mu(f)$, where $f$ is critically tame elliptic function
- If $\mathcal{S} = \{\phi_e\}_{e \in E}$ is a finitely irreducible strongly regular GDMS, then the metric entropy $h_{\tilde{\mu}_h}(\sigma)$ of the dynamical system $\sigma : E_A^{\mathbb{N}} \to E_A^{\mathbb{N}}$ with respect to the $\sigma$-invariant measure $\tilde{\mu}_h$ is finite.
- $h_{\mu_F}(F) = h_{\tilde{\mu}_h}(\sigma) < +\infty$, so $h_\mu(f) = h_{\mu_F}(F) \cdot \mu(J_c) < +\infty$.