On topological entropy:
when positivity implies $+\infty$

Sergiy Kolyada

dedicated to my Friend Lluis ALSEDA,
based on a joint work with Julia SEMIKINA

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Spain
Lubo, Lluis, Jaume and S. (Bellaterra, 1993)
On topological entropy: when positivity implies $+\infty$
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The main topic of our research is to study the relations between the properties of the topological semigroup $S(X)$ of all continuous maps from $X$ to $X$ (the topological group $H(X)$ of all homeomorphisms on $X$) and possible values of the topological entropy of its elements (continuous maps and homeomorphisms, respectively).
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1. Introduction and main results

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- when does a compact metric space admit a continuous map (homeomorphism) with positive topological entropy?

- when does the existence of a positive-entropy continuous map on a compact metric space imply the existence of a $+\infty$-entropy continuous map?
1. Introduction and main results

Theorem 1 (Groups representation)

Let $G$ be an arbitrary group. Then there exists a connected, locally connected, complete metric space $X$ (or alternatively compact, Hausdorff) for which the group of all autohomeomorphisms $H(X)$ is isomorphic to $G$.

Theorem 2 (Semigroups representation)

Let $S$ be a monoid (semigroup with identity). Then there exist:
1) a compact Hausdorff space $X$ such that the monoid of all nonconstant maps of $X$ into itself is isomorphic to $S$;
2) a connected metric space $X$ such that $S$ is isomorphic to the semigroup of all quasi-local homeomorphisms of $X$. 

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Let $X$, $Y$ be topological spaces. A mapping $f : X \to Y$ is called quasi-local homeomorphism if it is continuous and if for each open set $O \subset X$ there exists an open set $U \subset O$ such that $f|_U$ is a homeomorphism of $U$ onto $f(U)$.
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Aida B. Paalman-de Miranda, Johannes de Groot, Vera Trnkova
Given a compact metric space \((X, d)\), the set (group) \(H(X)\) of all self-homeomorphisms of \(X\), and the set (semigroup) \(S(X)\) of all continuous maps from \(X\) to \(X\), we will consider the following metrics on these sets:
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- the metric $d_u$ of uniform convergence (for $H(X)$):
  \[
  d_u(\varphi, \psi) = \sup_{x \in X} \max\{d(\varphi^{-1}(x), \psi^{-1}(x)), d(\varphi(x), \psi(x))\}.
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- the metric \(d_U\) of uniform convergence:
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d_U(\varphi, \psi) = \sup_{x \in X} d(\varphi(x), \psi(x)).
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The corresponding spaces will be denoted by \(H_U(X)\) and \(S_U(X)\). Note that \(d_U(\varphi, \psi)\) is well defined for any bounded selfmaps of \(X\) (in fact, for any selfmaps of \(X\), since \(X\) is bounded).
the Hausdorff metric $d_H$ (derived from the metric $d_{\max}((x_1, y_1), (x_2, y_2)) = \max\{d(x_1, x_2), d(y_1, y_2)\}$ in $X \times X$) applied to the graphs of maps (we identify a map and its graph, so we will write $d_H(\varphi, \psi)$). The corresponding spaces will be denoted by $H_H(X)$ and $S_H(X)$. 
Recall that the **Hausdorff distance** $d_H$ between two sets $A_1$ and $A_2$ in a metric space $X$ is given by

$$d_H(A_1, A_2) = \inf \{ \varepsilon > 0 : \overline{B}_\varepsilon[A_1] \supseteq A_2 \text{ and } \overline{B}_\varepsilon[A_2] \supseteq A_1 \},$$

where $\overline{B}_\varepsilon[A]$ denote the union of all closed balls of radius $\varepsilon > 0$ whose centers run over the elements of $A$. 
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where $\overline{B}_\varepsilon[A]$ denote the union of all closed balls of radius $\varepsilon > 0$ whose centers run over the elements of $A$.

This is a metric on the family of all bounded, nonempty closed subsets of $X$. Note that if $X$ is compact we can apply this metric to (the graphs of) continuous maps from $X$ to $X$. 
For any $\varphi, \psi \in H(X)$ we have $d_H(\varphi, \psi) \leq d_U(\varphi, \psi) \leq d_u(\varphi, \psi)$.
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For any \( \varphi, \psi \in H(X) \) we have \( d_H(\varphi, \psi) \leq d_U(\varphi, \psi) \leq d_u(\varphi, \psi) \) and for any \( \varphi, \psi \in S(X) \) we have \( d_H(\varphi, \psi) \leq d_U(\varphi, \psi) \). If \( X \) is a compact metric space then the topologies given by the uniform metric and Hausdorff metric are equivalent in \( H(X) \) and in \( S(X) \).
In particular, a subset of $H(X)$ (of $S(X)$) is compact in $H_u(X)$ (in $S_U(X)$) if and only if it is compact in $H_H(X)$ (in $S_H(X)$).
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In particular, a subset of $H(X)$ (of $S(X)$) is compact in $H_u(X)$ (in $S_U(X)$) if and only if it is compact in $H_H(X)$ (in $S_H(X)$). In general the two metrics are not uniformly equivalent (two maps which are close to each other in the Hausdorff metric, may have a large distance in the uniform metric). Of course, on compact subsets of $H(X)$ (of $S(X)$) they are uniformly equivalent.
Therefore $H_u(X) (S_U(X))$ is compact if and only if $H_H(X)$ ($S_H(X)$, respectively) is compact.
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Finally, recall that the space $H_u(X)$ ($S_U(X)$) is complete, but in general not totally bounded and the space $H_H(X)$ ($S_H(X)$) is totally bounded, but in general not complete and so in general none of them is compact.
Nevertheless, **the compactness** of the $S(X)$ and $H(X)$ is not a very strict condition and takes place quite often, because both of them may be 'very small'.
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Recall that a topological space $X$ is said to be rigid, if the full topological homeomorphism group $H(X)$ is the identity.

De Groot and Wille showed the existence of rigid plane locally connected one-dimensional continua (Peano curves).
1. Introduction and main results

The Peano curve is the disc $D$ with the interiors of all the propellers removed.

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1. Introduction and main results

Later, Cooke constructed two metric continua such that: the space of all continuous maps $S(X)$ (the full topological transformation semigroup) consists only of the constant maps and the identity; the space of all homeomorphisms $H(X)$ (the full topological transformation group) is topologically equivalent to the Cantor set.

But it is still an open problem what we can say about the topological structure of the compact group $H(X)$ and of the compact semigroup $S(X)$. 

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But it is still an open problem what can we say about the topological structure of the compact group $H(X)$ and of the compact semigroup $S(X)$.
Recall that a topological group is called \textit{profinite group} if it is Hausdorff, compact, and totally disconnected.
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Conjecture

Let $G$ be a compact group. Then the following conditions are equivalent:

1. There is a compact connected space $X$ such that $G \cong H(X)$.
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By a topological dynamical system we mean a pair \((X, f)\), where \(X\) is a metric compact space with metric \(d\) and \(f : X \to X\) is a continuous map.
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The concept of the **functional envelope** was first introduced by Auslander, Snoha and K.S.
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**Definition 1**

*By the **functional envelope** of a dynamical system $(X, f)$ we mean the dynamical system $(S(X), F_f)$, where the map $F_f : S(X) \to S(X)$ is defined by $F_f(\varphi) = f \circ \varphi$, $\varphi \in S(X)$.***
Remark

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The natural question arises here: what is the connection between the dynamical properties of $f$ and the corresponding map $F$, in particular between their topological entropies $h(f)$ and $h(F)$.
We will use Bowen-Dinaburg’s definition of the topological entropy. For $n \geq 1$ consider metrics $d_n$ which takes into account the distance between the respective $n$ initial iterates of points, namely put $d_n(x, y) = \max_{0 \leq i < n} d(f^i(x), f^i(y))$. 
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$$d_n(x, y) = \max_{0 \leq i < n} d(f^i(x), f^i(y)).$$
A subset $E$ of $X$ is called $(n, f, \varepsilon)$–separated if for every two different points $x, y \in E$ it holds $d_n(x, y) > \varepsilon$. 
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A subset $E$ of $X$ is called $(n, f, \varepsilon)$–separated if for every two different points $x, y \in E$ it holds $d_n(x, y) > \varepsilon$. For a compact set $K \subset X$ let $\text{sep}(n, \varepsilon, K)$ be the largest cardinality of any $(n, f, \varepsilon)$–separated set contained in $K$. 
Let \( f : X \to X \) be a uniformly continuous map. Then define

\[
h(f, K) = \lim_{\varepsilon \to 0} \limsup_{n \to \infty} \frac{1}{n} \log \text{sep}(n, \varepsilon, K)
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and **topological entropy of $f$** to be

$$h(f) = \sup_{K \subset X} h(f, K),$$

the supremum is taken over all compact subsets of $X$. 
Theorem A

Let $X$ be a compact metric space. If $S(X)$ is compact, then for any $f \in S(X)$ topological entropy of $(X, f)$ and topological entropy the functional envelope $(S(X), F_f)$ is zero.
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Let $X$ be a compact metric space. If $S(X)$ is compact, then for any $f \in S(X)$ topological entropy of $(X, f)$ and topological entropy the functional envelope $(S(X), F_f)$ is zero. If $H(X)$ is compact, then for any $f \in H(X)$ topological entropy of $(X, f)$ is also zero.
As it was mentioned before, it seems that all compact groups $H(X)$ are totally disconnected.
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For instance, the group $H(X)$ is totally disconnected for the following spaces
1. Introduction and main results

Sierpinski carpet
Menger-Schwamm universal curve
But is not compact, because these spaces admit homeomorphisms with positive topological entropy.
Now we will use again the term ‘continuum’ as a nonempty, compact, metrizable, connected space.
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A *chain* is a finite collection of open sets $\mathcal{C} = \{C_1, C_2, \ldots, C_n\}$ in a metric space such that $C_i \cap C_j \neq \emptyset$ if and only if $|i - j| \leq 1$. The elements of a chain are called its links, and a chain is called an $\varepsilon$-chain if each of its links has diameter less than $\varepsilon$. 

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A continuum $X$ is said to be chainable if, for every $\varepsilon > 0$, there exists an $\varepsilon$-chain covering $X$. 
1. Introduction and main results

A continuum is said to be indecomposable if it is not the union of two of its proper subcontinua, and hereditarily indecomposable if each of its subcontinua is indecomposable. The pseudo-arc is a nondegenerate, hereditarily indecomposable, chainable continuum.
A continuum is said to be *indecomposable* if it is not the union of two of its proper subcontinua, and *hereditarily indecomposable* if each of its subcontinua is indecomposable.
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The \textit{pseudo-arc} is a nondegenerate, hereditarily indecomposable, chainable continuum.
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2. In the hyperspace $C(X)$ of subcontinua of any Hilbert space or Euclidean space $X$ of dimension at least 2, the collection of continua which are pseudo-arcs is a dense $G_δ$. 
1. The pseudo-arc is a *homogeneous* continuum. **Every non-separating non-degenerate homogeneous plan continuum is a pseudo-arc!** (Very recent result of Hoehn and Oversteegen).

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3. The subset $\hat{H}(P)$ of maps of the pseudo-arc $P$ into itself which are homeomorphisms onto their images is a dense $G_\delta$ in $S(P)$. 
Question

Is the space of autohomeomorphisms of the pseudo-arc totally disconnected?
1. Introduction and main results

**Question**

*Is the space of autohomeomorphisms of the pseudo-arc totally disconnected?*

**Conjecture**

*Let $P$ be a pseudo-arc and $f \in S(P)$. Then the only possible values of topological entropy $h(f)$ are 0 and $+\infty$.*
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We will use now the term ‘continuum’ in two distinct meanings: the nondenumerable set of real numbers (set theory) and a nonempty, compact, metrizable, connected space (topology).
We will use now the term ‘continuum’ in two distinct meanings: the nondenumerable set of real numbers (set theory) and a nonempty, compact, metrizable, connected space (topology). A locally connected continuum is often called a Peano continuum.
It is well known that the topological entropy of a generic homeomorphism (continuous map) on a compact manifold with or without boundary the dimension of which is greater than one and the topological entropy of a generic continuous map on the interval is $+\infty$. 
In general we do not know when a compact metric space admits a continuous map (homeomorphism) with positive topological entropy and when this implies the existence of a continuous map (homeomorphism) with $+\infty$ topological entropy.
In general we do not know when a compact metric space admits a continuous map (homeomorphism) with positive topological entropy and when this implies the existence of a continuous map (homeomorphism) with $+\infty$ topological entropy. Nevertheless we have the following

**Theorem B**

Let $X$ be a compact metric space with a nontrivial Peano subcontinuum or with continuum many connected components. Then $X$ admits a continuous map with infinite topological entropy, i.e. there exists a continuous map $f : X \to X$ with $h(f) = +\infty$. 
Some examples of computing the topological entropy of the functional envelope \((S(X), F_f)\) was found by Auslander, Snoha and K.S.

This observation gives rise to the following conjecture.

**Conjecture**

Let \((X, f)\) be a topological dynamical system and \((S(X), F_f)\) its functional envelope. Then the only possible values of \(h(F_f)\) are 0 and \(+\infty\).
Some examples of computing the topological entropy of the functional envelope $(S(X), F_f)$ was found by Auslander, Snoha and K.S. and in all of them the topological entropy takes one of the values $\{0, +\infty\}$.
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**Conjecture**

Let \((X, f)\) be a topological dynamical system and \((S(X), F_f)\) its functional envelope. Then the only possible values of \(h(F_f)\) are 0 and \(+\infty\).
First progress on this conjecture was done by Matviichuk, who considered the case of the compact interval.
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First progress on this conjecture was done by Matviichuk, who considered the case of the compact interval. He proved that zero topological entropy of the system $(I, f)$, where $f$ is a continuous selfmap of the interval $I$, implies zero topological entropy of its functional envelope $(S_H(I), F_f)$ (and it was known that positivity of topological entropy of $(I, f)$ implies infinite entropy of $(S_H(I), F_f)$).
We describe a method of proving this conjecture for certain class of spaces, in particular it works in the case of an interval and the Cantor set.
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**Theorem C**

Let $X$ be one of the following compact metric spaces: a Peano continuum or a space with continuum many connected components. Let $f$ be a continuous selfmap of $X$. Then the only possible values of $h(F_f)$ are $0$ and $+\infty$. 
A sketch of the proof of Theorem B.
In order to prove it, we used the following generalisation of the commutativity of the entropy.
A sketch of the proof of Theorem B.
In order to prove it, we used the following generalisation of the commutativity of the entropy.

**Theorem 1**

Let $X$ and $Y$ be compact topological spaces. For any continuous maps $f : X \to Y$ and $g : Y \to X$ the topological entropies $h(g \circ f, X)$ and $h(f \circ g, Y)$ are equal.
Recall that a space $Y$ is said to be \textit{arcwise connected} provided that any two points of $Y$ can be joined by an arc in $Y$. It is known that every Peano continuum is arcwise connected.
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Let $P$ be a nontrivial Peano subcontinuum of the compact metric space $X = (X, d)$. Take a point $x_0 \in P$. Let us define the map $r : X \to I$ by $r(x) = d(x_0, x) \cdot (\max_{x \in X} d(x_0, x))^{-1}$, which is continuous and surjective.
Now, take a point $x_1 \in P$ and assume without loss of generality that $r(x_1) = 1$. Since $P$ is arcwise connected there is an arc $L \subset P$ which connects $x_0$ and $x_1$, i.e. a homeomorphism $\phi : I \to L$ such that $\phi(0) = x_0$ and $\phi(1) = x_1$, where $I = [0, 1]$. 
1. Introduction and main results

On topological entropy: when positivity implies $+\infty$
Now we define a continuous map $g : I \to I$ with topological entropy $h(g) = +\infty$ in a standard way.
On topological entropy: when positivity implies +infinity
So, we have defined the following continuous map $f := \varphi \circ g \circ r$ from $X$ into itself, more precisely from $X$ onto $L$. 
So, we have defined the following continuous map \( f := \varphi \circ g \circ r \) from \( X \) into itself, more precisely from \( X \) onto \( L \). We can rewrite this composition as the following (so called) non-autonomous dynamical system.
So, we have defined the following continuous map $f := \varphi \circ g \circ r$ from $X$ into itself, more precisely from $X$ onto $L$. We can rewrite this composition as the following (so called) non-autonomous dynamical system

\[
X \xrightarrow{r} I \xrightarrow{g} I \xrightarrow{\varphi} L \xrightarrow{r} I \xrightarrow{g} I \xrightarrow{\varphi} L \xrightarrow{r} I \xrightarrow{g} I \xrightarrow{\varphi} L \xrightarrow{r} I \xrightarrow{g} I \xrightarrow{\varphi} L \xrightarrow{r} I \xrightarrow{g} I \xrightarrow{\varphi} L \xrightarrow{r} \ldots.
\]
Since we have the following inequality for topological entropy
\[ h(f) \geq h(f|_L, L) = h(\phi \circ g \circ r, L), \]
Since we have the following inequality for topological entropy
\[ h(f) \geq h(f|_L, L) = h(\varphi \circ g \circ r, L), \]
instead it we will consider the following periodic non-autonomous dynamical system

\[ L \overset{r}{\rightarrow} I \overset{g}{\rightarrow} L \overset{\varphi}{\rightarrow} L \overset{r}{\rightarrow} I \overset{g}{\rightarrow} L \overset{\varphi}{\rightarrow} L \overset{r}{\rightarrow} I \overset{g}{\rightarrow} L \overset{\varphi}{\rightarrow} \ldots \overset{r}{\rightarrow} L \overset{g}{\rightarrow} I \overset{\varphi}{\rightarrow} L \overset{r}{\rightarrow} I \overset{g}{\rightarrow} ... \]
Using the commutativity of topological entropy, we can write this as \( h(f|_L, L) = h(\varphi \circ g \circ r, L) = h(r \circ \varphi \circ g, I) = h(g \circ r \circ \varphi, I). \)
Using the commutativity of topological entropy, we can write this as $h(f|_L, L) = h(\varphi \circ g \circ r, L) = h(r \circ \varphi \circ g, I) = h(g \circ r \circ \varphi, I)$.

Since $\varphi(I_n) = L_n$ and $r(L_n) = I_n$ for any $n \geq 1$, the map $g \circ r \circ \varphi : I \rightarrow I$ on the interval $I_n$ has an $s(n)$-horseshoe for any $n \geq 1$, where $s(n) \geq 2n + 1$. 
Using the commutativity of topological entropy, we can write this as \( h(f|_L, L) = h(\varphi \circ g \circ r, L) = h(r \circ \varphi \circ g, I) = h(g \circ r \circ \varphi, I). \)

Since \( \varphi(I_n) = L_n \) and \( r(L_n) = I_n \) for any \( n \geq 1 \), the map \( g \circ r \circ \varphi : I \overset{onto}{\rightarrow} I \) on the interval \( I_n \) has an \( s(n) \)-horseshoe for any \( n \geq 1 \), where \( s(n) \geq 2n + 1 \).

Therefore topological entropy

\[
h(f) \geq h(f|_L, L) = h(g \circ r \circ \varphi, I) = +\infty.
\]
The proof of Theorem A.
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Recall that $S_U(X)$ is compact if and only if $S_H(X)$ is compact. So, let us prove the theorem for $S(X) = S_U(X)$. 

The proof of Theorem A.

Recall that $S_U(X)$ is compact if and only if $S_H(X)$ is compact. So, let us prove the theorem for $S(X) = S_U(X)$. Since space $S_U(X)$ is compact, by the Ascoli theorem $S_U(X)$ is an equicontinuous family of maps on $X$. 
It means that for every $\varepsilon > 0$ there is a $\delta > 0$ such that for all $x, y \in X$ with $d(x, y) < \delta$ and all $\varphi \in SU(X)$ we have $d(\varphi(x), \varphi(y)) < \varepsilon$. 
It means that for every $\varepsilon > 0$ there is a $\delta > 0$ such that for all $x, y \in X$ with $d(x, y) < \delta$ and all $\varphi \in S_U(X)$ we have $d(\varphi(x), \varphi(y)) < \varepsilon$.

In particular, if $d_U(\varphi_1, \varphi_2) < \delta$, $\varphi_1, \varphi_2 \in S_U(X)$, then for every $g \in S_U(X)$ it holds $d_U(g \circ \varphi_1, g \circ \varphi_2) < \varepsilon$. 
It means that for every $\varepsilon > 0$ there is a $\delta > 0$ such that for all $x, y \in X$ with $d(x, y) < \delta$ and all $\varphi \in S_U(X)$ we have $d(\varphi(x), \varphi(y)) < \varepsilon$.

In particular, if $d_U(\varphi_1, \varphi_2) < \delta$, $\varphi_1, \varphi_2 \in S_U(X)$, then for every $g \in S_U(X)$ it holds $d_U(g \circ \varphi_1, g \circ \varphi_2) < \varepsilon$. Hence the map $F_f$ is equicontinuous on the space $S_U(X)$ (i.e., the family \{ $F_f, F_f^2, ..., F_f^n, ...$ \} is equicontinuous).
If a set is \((n, F_f, \varepsilon)\)-separated, then it cannot contain points from 
\(S_U(X)\) on a distance less than a corresponding \(\delta\).
If a set is \((n, F_f, \varepsilon)\)-separated, then it cannot contain points from \(S_U(X)\) on a distance less than a corresponding \(\delta\). Since \(S_U(X)\) is compact, the cardinalities of such sets are bounded above by some constant depending only on \(\delta\).
If a set is \((n, F_f, \varepsilon)\)-separated, then it cannot contain points from \(S_U(X)\) on a distance less than a corresponding \(\delta\). Since \(S_U(X)\) is compact, the cardinalities of such sets are bounded above by some constant depending only on \(\delta\) (in fact depending on \(\varepsilon\)):

\[
\text{sep} (n, F_f, \varepsilon) \leq C(\varepsilon).
\]
It follows

\[ h(F_f) = \lim_{\varepsilon \to 0} \lim_{n \to \infty} \frac{1}{n} \log \text{sep}(n, F_f, \varepsilon) \leq \lim_{\varepsilon \to 0} \lim_{n \to \infty} \frac{1}{n} \log C(\varepsilon) = 0. \]
It follows

\[ h(F_f) = \lim_{\varepsilon \to 0} \limsup_{n \to \infty} \frac{1}{n} \log \text{sep}(n, F_f, \varepsilon) \leq \lim_{\varepsilon \to 0} \limsup_{n \to \infty} \frac{1}{n} \log C(\varepsilon) = 0. \]

Therefore \( 0 = h_U(F) \geq h_H(F) \geq h(f) \).
As we mentioned before the case of a functional envelope $(H(X), F_f)$ if $f \in H(X)$ is a bit more complicated.
As we mentioned before the case of a functional envelope \((H(X), F_f)\) if \(f \in H(X)\) is a bit more complicated. Formally there is no direct relation between topological entropies \(h(F_f)\) and \(h(f)\) in this case.
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Since $H(X)$ is compact, the $\omega$-limit set $\omega_{F_f}(\text{id}_X)$ is nonempty.
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Since $H(X)$ is compact, the $\omega$-limit set $\omega_{F_f}(\text{id}_X)$ is nonempty. By a proposition of the Auslander, Snoha and K. it follows that $(X, f)$ is uniformly rigid and therefore topological entropy $h(f) = 0$. 