On topological entropy: when positivity implies +infinity

Sergiy Kolyada

dedicated to my Friend Lluis ALSEDA,
based on a joint work with Julia SEMIKINA

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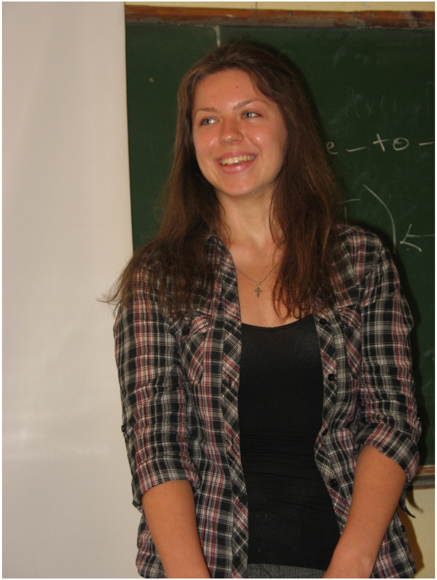
Spain

Lubo, Lluis, Jaume and S. (Bellaterra, 1993)



JULIA SEMIKINA

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- **when does the existence of a positive-entropy continuous map on a compact metric space imply the existence of a** $+\infty$**-entropy continuous map**?

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Let G be an arbitrary group. Then there exists a connected, locally connected,complete metric space X (or alternatively compact, Hausdorff) for which the group of all autohomeomorphisms $H(X)$ is isomorphic to G.

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1) *a compact Hausdorff space X such that the monoid of all nonconstant maps of X into itself is isomorphic to S;*
2) *a connected metric space X such that S is isomorphic to the semigroup of all quasi-local homeomorphisms of X.*

Let X, Y be topological spaces. A mapping $f : X \to Y$ is called quasi-local homeomorphism if it is continuous and if for each opene set $O \subset X$ there exists an open set $U \subset O$ such that $f|_U$ is a homeomorphism of U onto $f(U)$.

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Aida B. Paalman-de Miranda, Johannes de Groot, Vera Trnkova

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- the metric d_u of uniform convergence (for $H(X)$): $d_u(\varphi, \psi) = \sup_{x \in X} \max\{d(\varphi^{-1}(x), \psi^{-1}(x)), d(\varphi(x), \psi(x))\}$. The corresponding space will be denoted by $H_u(X)$.

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- the metric d_U of uniform convergence: $d_U(\varphi, \psi) = \sup_{x\in X} d(\varphi(x), \psi(x))$. The corresponding spaces will be denoted by $H_U(X)$ and $S_U(X)$. Note that $d_U(\varphi, \psi)$ is well defined for any bounded selfmaps of X (in fact, for any selfmaps of X, since X is bounded).

- the Hausdorff metric d_H (derived from the metric $d_{\max}((x_1, y_1), (x_2, y_2)) = \max\{d(x_1, x_2), d(y_1, y_2)\}$ in $X \times X$) applied to the graphs of maps (we identify a map and its graph, so we will write $d_H(\varphi, \psi)$). The corresponding spaces will be denoted by $H_H(X)$ and $S_H(X)$.

Recall that the ***Hausdorff distance*** d_H between two sets A_1 and A_2 in a metric space X is given by

$$d_H(A_1, A_2) = \inf\{\varepsilon > 0 : \ \overline{B}_\varepsilon[A_1] \supseteq A_2 \text{ and } \overline{B}_\varepsilon[A_2] \supseteq A_1\} \ ,$$

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where $\overline{B}_\varepsilon[A]$ denote the union of all closed balls of radius $\varepsilon > 0$ whose centers run over the elements of A.

This is a metric on the family of all bounded, nonempty closed subsets of X. Note that if X is compact we can apply this metric to (the graphs of) continuous maps from X to X.

For any $\varphi, \psi \in H(X)$ we have $d_H(\varphi, \psi) \leq d_U(\varphi, \psi) \quad \leq d_u(\varphi, \psi)$

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In particular, a subset of $H(X)$ (of $S(X)$) is compact in $H_u(X)$ (in $S_U(X)$) if and only if it is compact in $H_H(X)$ (in $S_H(X)$).

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Finally, recall that **the space** $H_u(X)$ $(S_U(X))$ **is complete**, but in general not totally bounded and **the space** $H_H(X)$ $(S_H(X))$ **is totally bounded**, but in general not complete and so **in general none of them is compact**.

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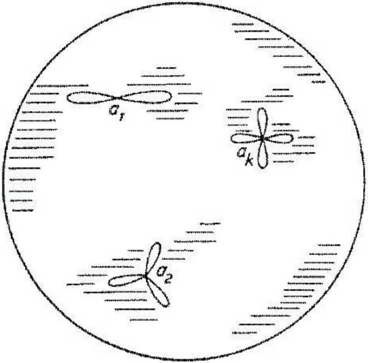
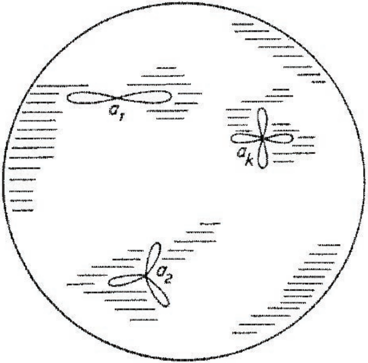
De Groot and Wille showed the existence of rigid plane locally connected one-dimensional continua (*Peano curves*).


a_1
a_k
a_2



The Peano curve is the disc D with the interiors of all the propellers removed.

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But it is still an open problem what can we say about the topological structure of the compact group $H(X)$ and of the compact semigroup $S(X)$.

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Conjecture

Let G be a compact group. Then the following conditions are equivalent:

(1) *There is a compact connected space X such that $G \cong H(X)$.*

(2) *There is a compact space X such that $G \cong H(X)$.*

(3) *G is profinite.*

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Definition 1

*By the **functional envelope** of a dynamical system (X, f) we mean the dynamical system $(S(X), F_f)$, where the map $F_f : S(X) \to S(X)$ is defined by $F_f(\varphi) = f \circ \varphi,\ \varphi \in S(X)$.*

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The functional envelope of a system always contains a copy of the original system, namely the constant mappings. Hereby we easily conclude that $h(F) \geq h(f)$.

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The natural question arises here: what is the connection between the dynamical properties of f and the corresponding map F, in particular between their topological entropies $h(f)$ and $h(F)$.

We will use Bowen-Dinaburg's definition of the topological entropy. For $n \geq 1$ consider metrics d_n which takes into account the distance between the respective n initial iterates of points, namely put $d_n(x, y) = \max_{0 \leq i < n} d(f^i(x), f^i(y))$.

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A subset E of X is called (n, f, ε)–*separated* if for every two different points $x, y \in E$ it holds $d_n(x, y) > \varepsilon$.

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A subset E of X is called (n, f, ε)–*separated* if for every two different points $x, y \in E$ it holds $d_n(x,y) > \varepsilon$. For a compact set $K \subset X$ let $sep(n, \varepsilon, K)$ be the largest cardinality of any (n, f, ε)–separated set contained in K.

Let $f : X \to X$ be a uniformly continuous map. Then define

$$h(f, K) = \lim_{\varepsilon \to 0} \limsup_{n \to \infty} \frac{1}{n} \log sep(n, \varepsilon, K)$$

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and **topological entropy of** f to be

$$h(f) = \sup_{K \subset X} h(f, K),$$

the supremum is taken over all compact subsets of X.

Theorem A

Let X be a compact metric space. If $S(X)$ is compact, then for any $f \in S(X)$ topological entropy of (X, f) and topological entropy the functional envelope $(S(X), F_f)$ is zero.

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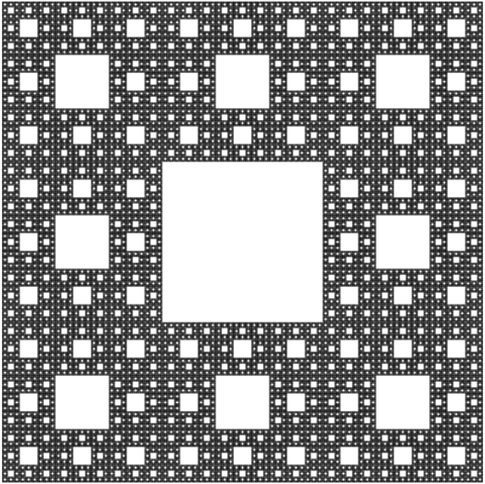
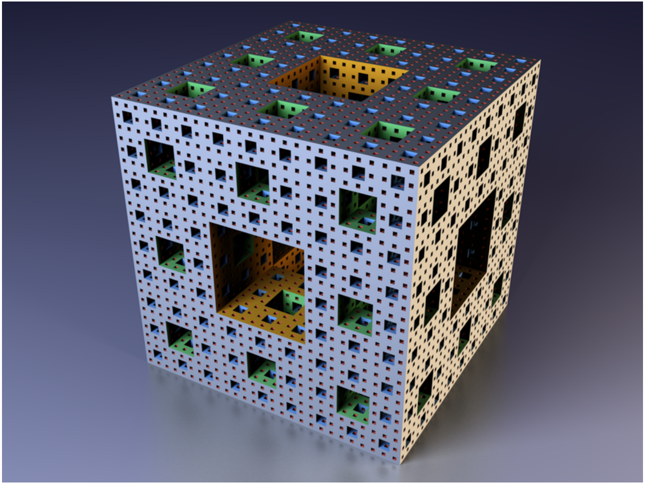
For instance, the group $H(X)$ is totally disconnected for the following spaces



Sierpinski carpet



Menger-Schwamm universal curve

But is not compact, becase these spaces admit homeomorphisms with positive topological entropy.

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A *chain* is a finite collection of open sets $\mathcal{C} = \{C_1, C_2, \dots, C_n\}$ in a metric space such that $C_i \cap C_j \neq \emptyset$ if and only if $|i - j| \leq 1$. The elements of a chain are called its links, and a chain is called an ε-chain if each of its links has diameter less than ε.

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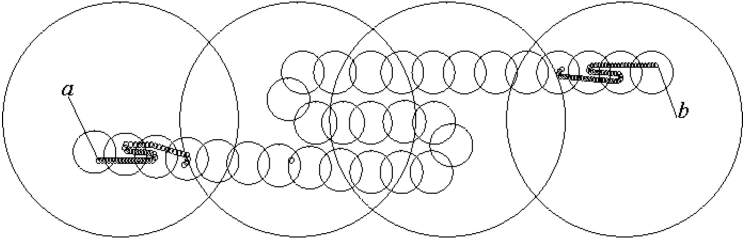
A continuum X is said to be *chainable* if, for every $\varepsilon > 0$, there exists an ε-chain covering X.


a
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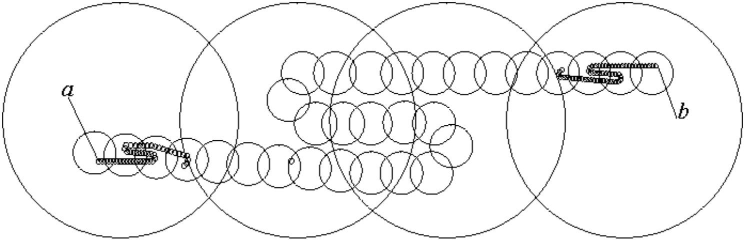
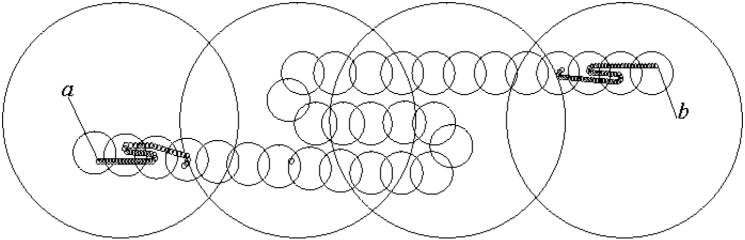


A continuum is said to be *indecomposable* if it is not the union of two of its proper subcontinua, and *hereditarily indecomposable* if each of its subcontinua is indecomposable.



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3. The subset $\hat{H}(P)$ of maps of the pseudo-arc P into itself which are homeomorphisms onto their images is a dense G_δ in $S(P)$.

Question

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Conjecture

Let P be a pseudo-arc and $f \in S(P)$. Then the only possible values of topological entropy $h(f)$ are 0 and $+\infty$.

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We will use now the term '***continuum***' in two distinct meanings: the nondenumerable set of real numbers (set theory) and a nonempty, compact, metrizable, connected space (topology). A locally connected continuum is often called a *Peano continuum*.

It is well known that the topological entropy of a generic homeomorphism (continuous map) on a compact manifold with or without boundary the dimension of which is greater then one and the topological entropy of a generic continuous map on the interval is $+\infty$

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Theorem B

Let X be a compact metric space with a nontrivial Peano subcontinuum or with continuum many connected components. Then X admits a continuous map with infinite topological entropy, i.e. there exists a continuous map $f : X \to X$ with $h(f) = +\infty$.

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Conjecture

Let (X, f) be a topological dynamical system and $(S(X), F_f)$ its functional envelope. Then the only possible values of $h(F_f)$ are 0 and $+\infty$.

First progress on this conjecture was done by Matviichuk, who considered the case of the compact interval.

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We describe a method of proving this conjecture for certain class of spaces, in particular it works in the case of an interval and the Cantor set.

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Theorem C

Let X be one of the following compact metric spaces: a Peano continuum or a space with continuum many connected components. Let f be a continuous selfmap of X. Then the only possible values of $h(F_f)$ are 0 and $+\infty$.

A sketch of the proof *of Theorem B.*
In order to prove it, we used the following generalisation of the commutativity of the entropy.

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Theorem 1

Let X and Y be compact topological spaces. For any continuous maps $f : X \to Y$ and $g : Y \to X$ the topological entropies $h(g \circ f, X)$ and $h(f \circ g, Y)$ are equal.

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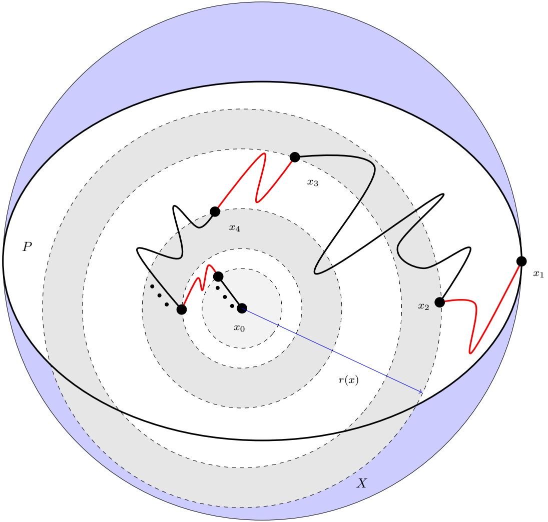
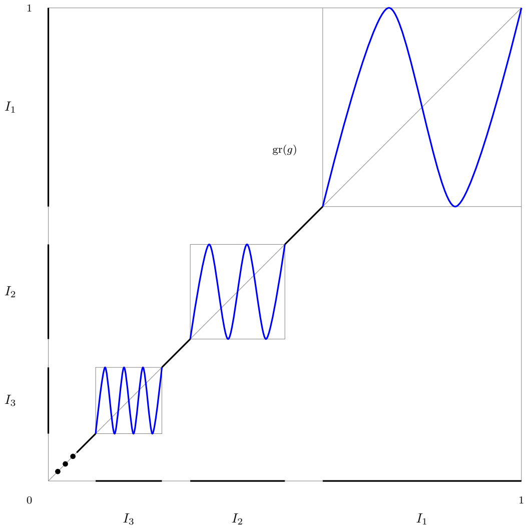
Let P be a nontrivial Peano subcontinuum of the compact metric space $X = (X, d)$. Take a point $x_0 \in P$. Let us define the map $r : X \to I$ by $r(x) = d(x_0, x) \cdot (\max_{x \in X} d(x_0, x))^{-1}$, which is continuous and surjective.

Now, take a point $x_1 \in P$ and assume without loss of generality that $r(x_1) = 1$. Since P is arcwise connected there is an arc $L \subset P$ which connects x_0 and x_1, i. e. a homeomorphism $\phi : I \to L$ such that $\phi(0) = x_0$ and $\phi(1) = x_1$, where $I = [0, 1]$.



x_3
x_4
P
x_1
x_2
x_0
$r(x)$
X

Now we define a continuous map $g : I \to I$ with topological entropy $h(g) = +\infty$ in a standard way.


1
I_1
gr(g)
I_2
I_3
0
1
I_3
I_2
I_1

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$$X \xrightarrow{r} I \xrightarrow{g} I \xrightarrow{\varphi} L \xrightarrow{r} I \xrightarrow{g} I \xrightarrow{\varphi} L \xrightarrow{r} I \xrightarrow{g} \ldots \xrightarrow{\varphi} L \xrightarrow{r} I \xrightarrow{g} I \xrightarrow{\varphi} L \xrightarrow{r} \ \ldots \ .$$

Since we have the following inequality for topological entropy
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Since we have the following inequality for topological entropy $h(f) \geq h(f|_L, L) = h(\varphi \circ g \circ r, L)$, instead it we will consider the the following periodic non-autonomous dynamical system

$$L \xrightarrow{r} I \xrightarrow{g} I \xrightarrow{\varphi} L \xrightarrow{r} I \xrightarrow{g} I \xrightarrow{\varphi} L \xrightarrow{r} I \xrightarrow{g} \ldots \xrightarrow{\varphi} L \xrightarrow{r} I \xrightarrow{g} I \xrightarrow{\varphi} L \xrightarrow{r} \ \ldots \ .$$

Using the commutativity of topological entropy, we can write this as $h(f|_L, L) = h(\varphi \circ g \circ r, L) = h(r \circ \varphi \circ g, I) = h(g \circ r \circ \varphi, I)$.

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Since $\varphi(I_n) = L_n$ and $r(L_n) = I_n$ for any $n \geq 1$, the map $g \circ r \circ \varphi : I \overset{onto}{\to} I$ on the interval I_n has an $s(n)$-horseshoe for any $n \geq 1$, where $s(n) \geq 2n + 1$.

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Therefore topological entropy

$$h(f) \geq h(f|_L, L) = h(g \circ r \circ \varphi, I) = +\infty.$$

The proof *of Theorem A.*

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Recall that $S_U(X)$ is compact if and only if $S_H(X)$ is compact. So, let us prove the theorem for $S(X) = S_U(X)$.

The proof *of Theorem A.*

Recall that $S_U(X)$ is compact if and only if $S_H(X)$ is compact. So, let us prove the theorem for $S(X) = S_U(X)$. Since space $S_U(X)$ is compact, by the Ascoli theorem $S_U(X)$ is an equicontinuous family of maps on X.

It means that for every $\varepsilon > 0$ there is a $\delta > 0$ such that for all $x, y \in X$ with $d(x, y) < \delta$ and all $\varphi \in S_U(X)$ we have $d(\varphi(x), \varphi(y)) < \varepsilon$.

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In particular, if $d_U(\varphi_1, \varphi_2) < \delta,\ \varphi_1, \varphi_2 \in S_U(X)$, then for every $g \in S_U(X)$ it holds $d_U(g \circ \varphi_1, g \circ \varphi_2) < \varepsilon$.

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In particular, if $d_U(\varphi_1, \varphi_2) < \delta,\ \varphi_1, \varphi_2 \in S_U(X)$, then for every $g \in S_U(X)$ it holds $d_U(g \circ \varphi_1, g \circ \varphi_2) < \varepsilon$. Hence the map F_f is equicontinuous on the space $S_U(X)$ (i.e., the family $\{F_f, F_f^2, ..., F_f^n, ...\}$ is equicontinuous).

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If a set is (n, F_f, ε)-separated, then it cannot contains points from $S_U(X)$ on a distance less than a corresponding δ. Since $S_U(X)$ is compact, the cardinalities of such sets are bounded above by some constant depending only on δ (in fact depending on ε): $\mathrm{sep}\,(n, F_f, \varepsilon) \leq C(\varepsilon)$.

It follows

$$h(F_f) = \lim_{\varepsilon\to 0} \limsup_{n\to\infty} \frac{1}{n} \log \mathrm{sep}\,(n, F_f, \varepsilon) \leq \lim_{\varepsilon\to 0} \limsup_{n\to\infty} \frac{1}{n} \log C(\varepsilon) = 0.$$

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Therefore $0 = h_U(F) \geq h_H(F) \geq h(f)$.

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Since $H(X)$ is compact, the ω-limit set $\omega_{F_f}(\mathrm{id}_X)$ is nonempty. By a proposition of the Auslander, Snoha and K. it follows that (X, f) is uniformly rigid and therefore topological entropy $h(f) = 0$.