Manifolds on the verge of a hyperbolicity breakdown

Breakdown of normally hyperbolic invariant tori with qp dynamics

Àlex Haro

Departament de Matemàtica Aplicada i Anàlisi
Universitat de Barcelona

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Introduction

Persistence of invariant manifolds

- **The long term behavior of dynamical systems is organized by the invariant objects.**
- **It is important to understand which invariant objects persist under modifications of the system.**
- An invariant manifold persists under perturbations if and only if it is normally hyperbolic.
  [HirschP69][Fenichel71][Mañé78]
- There are spectral characterizations of hyperbolicity.
  [Mather68][HirschPS77][Swanson83]

Introduction

Destruction of invariant tori with quasi-periodic dynamics

In this talk:

- Continuation of invariant tori with respect to parameters, with a prescribed (Diophantine) frequency.
- Need of adjustment of parameters.
- Phenomena that happen at the breakdown of exponential dichotomies (loss of reducibility).
- Quantitative laws. (Empirically conjectured scaling properties.)

Invariant tori

Invariance equation

- Let us consider a family of maps
$$F_{a,\varepsilon} : \mathbb{T}^d \times \mathbb{R}^n \to \mathbb{T}^d \times \mathbb{R}^n,$$
where:
  - $a \in \mathbb{R}^d$ is the adjusting parameter;
  - $\varepsilon \in \mathbb{R}$ is the perturbation parameter;

  and a (Diophantine) frequency vector $\omega \in \mathbb{R}^d$.
- The dimension of the phase space is $m = d + n$.
- For ε fixed, a solution (K, a), where $K : \mathbb{T}^d \to \mathbb{T}^d \times \mathbb{R}^n$, of the equation
$$F_{a,\varepsilon}(K(\theta)) = K(\theta + \omega) \ , \tag{1}$$
parameterizes an invariant torus for $F_{a,\varepsilon}$
$$\mathcal{K} = \{K(\theta) \mid \theta \in \mathbb{T}^d\},$$
whose dynamics is the rotation by ω.

Invariant tori

Linearization

The linearization around the torus $K : \mathbb{T}^d \to \mathbb{T}^d \times \mathbb{R}^n$, $M(\theta) = \mathrm{D}F_{a,\varepsilon}(K(\theta))$, induces:

- a linear skew product (cocycle) in $\mathbb{R}^{d+n}$ over $\mathbb{T}^d$,

$$\left\{ \begin{array}{l} \bar{v} = M(\theta)v \\ \bar{\theta} = \theta + \omega \end{array} \right. ; \tag{2}$$

- a transfer operator $\mathcal{M}_\omega$ acting on bounded sections $\mathcal{v} : \mathbb{T}^d \to \mathbb{C}^{d+n}$ by

$$\mathcal{M}_\omega \mathcal{v}(\theta) = M(\theta - \omega)\mathcal{v}(\theta - \omega) \ . \tag{3}$$

The functional analysis properties (3) *are closely related to the dynamical properties of* (2).

Mather, Sacker, Sell, Palmer, Hirsch, Pugh, Shub, Mañé, Chicone, Swanson, Johnson, Latushkin, Stëpin, de la Llave, ...

Invariant tori

Spectrum and invariant bundles

Theorem (Spectral Theorem)

There is a spectral gap in the annulus of radii $0 < \lambda_- < \lambda_+$ *if and only if there is an invariant and continuous splitting* $\mathbb{R}^n = E_\theta^- \oplus E_\theta^+$ *characterized by the rates of growth*

$$\begin{array}{lcl} v \in E_\theta^- & \Leftrightarrow & |M^{+k}(\theta)v| \leq C(\lambda_-)^{+k}|v|, \;\; k \geq 0; \\ v \in E_\theta^+ & \Leftrightarrow & |M^{-k}(\theta)v| \leq C(\lambda_+)^{-k}|v|, \;\; k \geq 0. \end{array} \tag{4}$$

(exponential dichotomy)

- The spectrum of $\mathcal{M}_\omega$ is a set of annuli, centered at 0.
- The spectrum of the transfer operator acting on spaces of continuous and C^r sections is the same.

Invariant tori

Normal hyperbolicity

The torus is normally hyperbolic if the spectrum of the transfer operator has three spectral components:

- *the central component, the unit circle, which corresponds to the tangent bundle;*
- *the stable component inside the unit circle, producing the stable bundle;*
- *the unstable component outside the unit circle, producing the unstable bundle.*

Invariant tori

Reducibility

- The torus is reducible if its linearization $M(\theta)$ is reducible to constants, i.e.

$$M(\theta)P(\theta) = P(\theta + \omega)\Lambda \tag{5}$$

  for suitable $P(\theta)$ and constant matrix Λ.
- In such a case, the spectrum is a set of circles, one for each eigenvalue of Λ.
- The modulus of the eigenvalues are the Lyapunov multipliers.
- Reducibility is a desirable property. Unfortunately, it does not always hold.

A fattened Hénon map

The equations of the model

$$\begin{cases} \bar{\theta} = \theta + a + d\varepsilon(\cos(2\pi\theta) + y) \qquad (\text{mod } 1) \\ \bar{x} = 1 + y - b\ x^2 + \varepsilon \cos(2\pi\theta) \\ \bar{y} = cx \end{cases}$$

- a is the adjusting parameter, to obtain invariant tori with fixed frequency $\omega = \frac{1}{2}(\sqrt{5} - 1)$;
- b is the nonlinear parameter ($b = 0.68$);
- c is the dissipative parameter ($c = 0.1$);
- d is the coupling parameter;
- ε is the perturbation parameter.

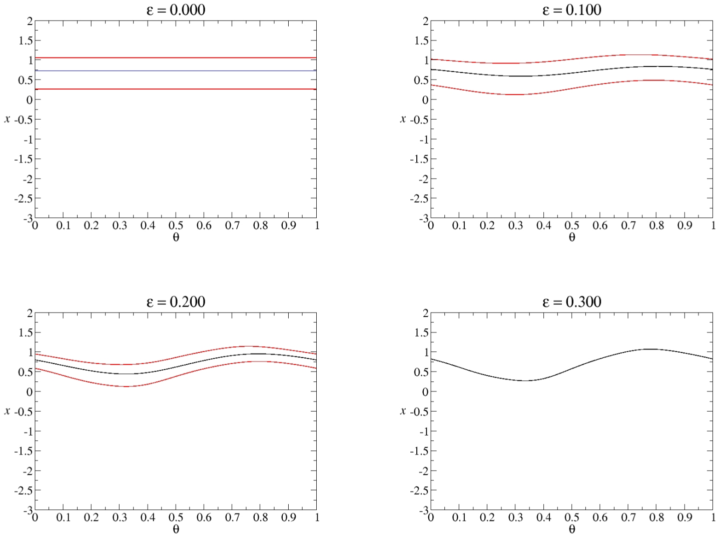
For $d = 0$: [Krauskopf,Osinga 98][Feudel,Osinga 00][Haro,de la Llave 06]

The adjusting parameter is constant $a = \omega$. This is the case we consider.

For $d \neq 0$: [Canadell, Haro]

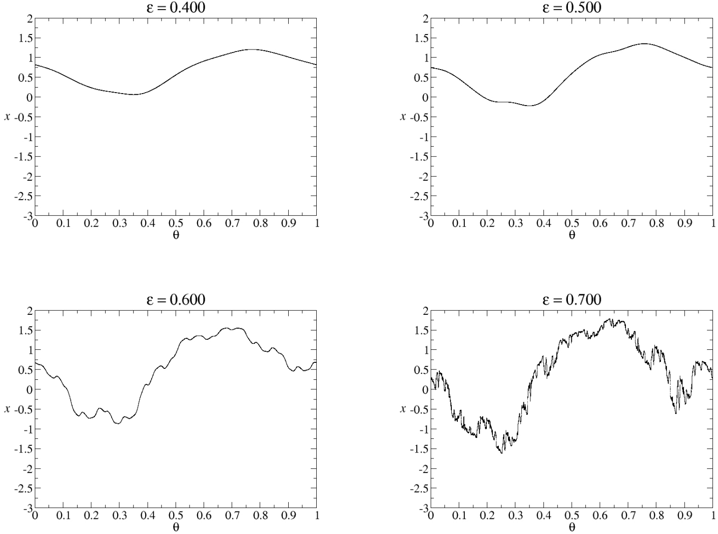
Continuation of an invariant torus

(I) Period "halving" (from saddle to attracting-node)



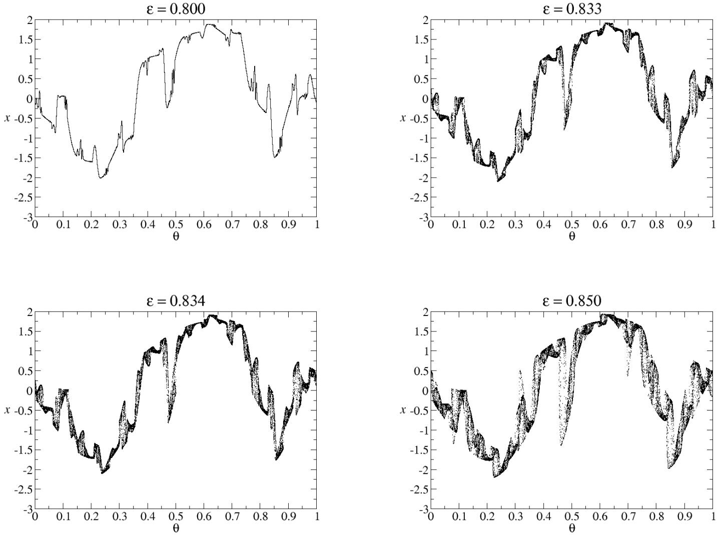
Continuation of an invariant torus

(II) Continuation of an attracting torus



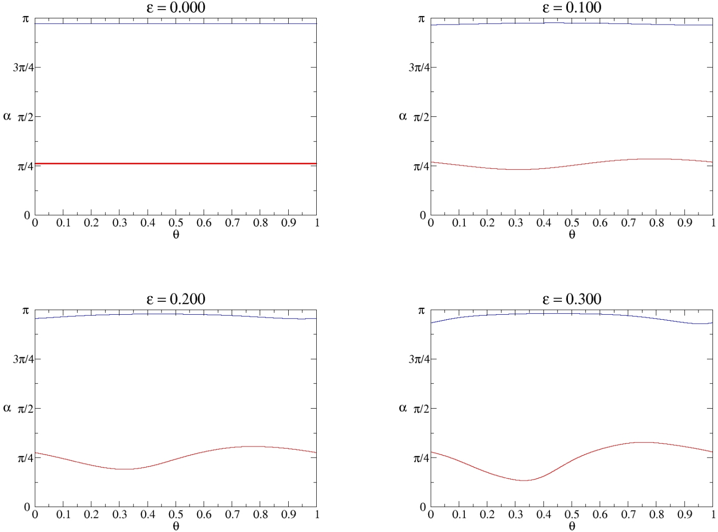
Continuation of an invariant torus

(III) Fractalization of the torus



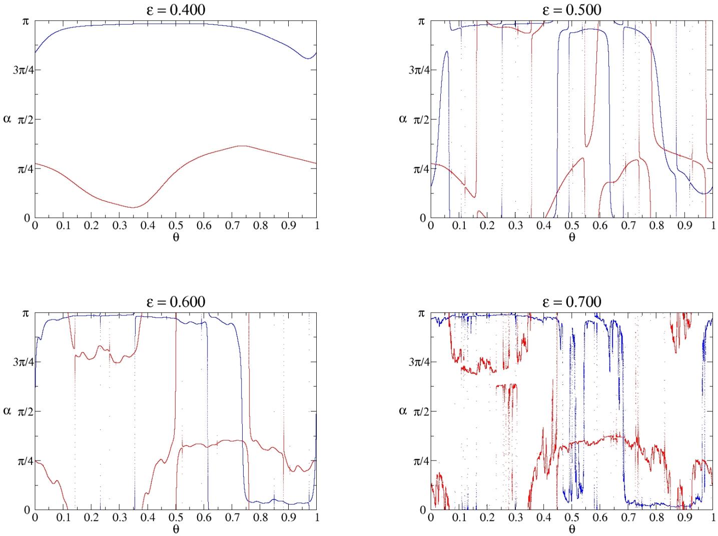
Invariant bundles (projectivization)

(I) Unstable bundle becomes a slow stable bundle



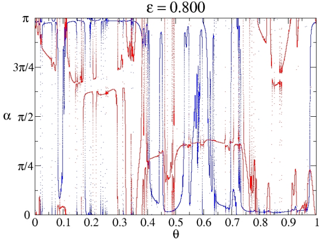
Invariant bundles (projectivization)

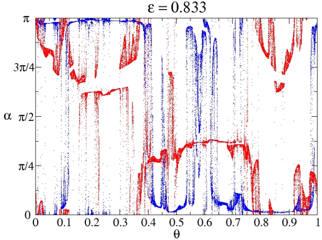
(II) Merging of bundles (collision of curves, SNA) (See also [Jalnine,Osbaldestin 05])

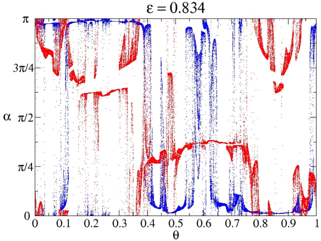


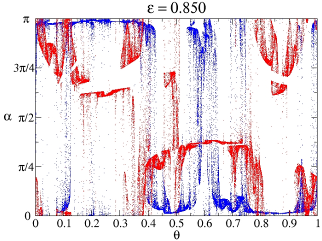
Invariant directions (projectivized bundles)

(III) Invariant directions for the fractalization of the torus



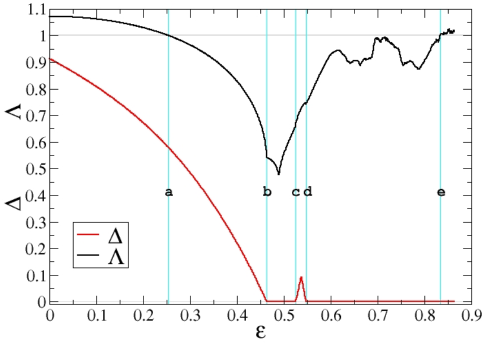






Description of the bifurcations

Observables: Λ (maximal Lyapunov multiplier)
Δ (distance beween bundles)



a) Period halving bifurcation.

b,c,d) Bundle merging bifurcation, SNAs in the projective dynamics.

e) Fractalization of the torus, a phenomenon not well understood.

The bundle merging bifurcation

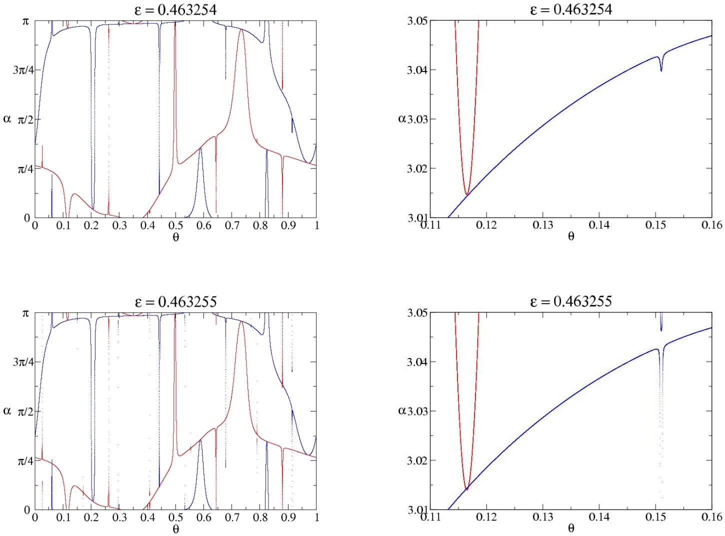
An obstruction to reducibility

Rotating Hénon map: b= 0.68, c= 0.1

ε	eigenvalues	error	nfm
0.000	-1.0721039594 , 0.0932745366	9.6e-21	100
0.200	-1.0297559933 , 0.0971103841	8.3e-21	100
0.400	-0.8288693291 , 0.1206462786	9.6e-20	100
0.450	-0.6721643269 , 0.1487731437	9.9e-13	100
0.460	-0.6034304995 , 0.1657191675	2.9e-14	300
0.461	-0.5925812920 , 0.1687532181	2.7e-12	300
0.462	-0.5792054526 , 0.1726503084	2.3e-13	400
0.463	-0.5584521519 , 0.1790663706	9.1e-10	6800

The bundle merging bifurcation

Visual verification (zooms)



The bundle merging bifurcation

An analytical/topological justification of bundle collapse

- For $\varepsilon = 0.460$, the torus is attracting and the cocycle is reducible to a constant diagonal matrix

$$\mathrm{diag}(-0.6034304995, 0.1657191675).$$

- For $\varepsilon = 0.530$, the torus is attracting and the cocycle is reducible to a constant diagonal matrix

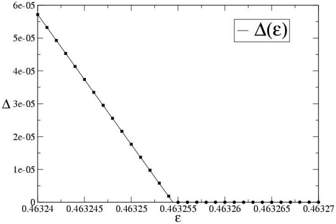
$$\mathrm{diag}(0.6945467500, -0.1439787890).$$

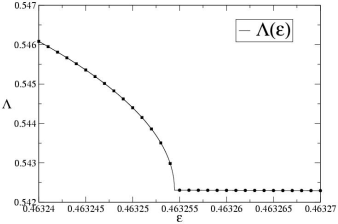
- Since the Lyapunov multipliers are different during the continuation,

  the cocycle can not be reducible during the whole continuation!

The bundle merging bifurcation

Quantitative estimates (universal laws)





$$\begin{cases} \Delta_\varepsilon \sim \alpha(\varepsilon_b - \varepsilon)^\beta & \text{if } \varepsilon \leq \varepsilon_b \\ \Delta_\varepsilon \approx 0 & \text{if } \varepsilon \geq \varepsilon_b \end{cases}$$

$$\begin{cases} \Lambda_\varepsilon \sim \Lambda_b + A(\varepsilon_b - \varepsilon)^B & \text{if } \varepsilon \leq \varepsilon_b \\ \Lambda_\varepsilon \approx \Lambda_b + \bar{A}(\varepsilon - \varepsilon_b)^{\bar{B}} & \text{if } \varepsilon \geq \varepsilon_b \end{cases}$$

$\varepsilon_b = 0.46325447112$
$\alpha = 3.94933$
$\beta = 0.999979 \approx 1$

Bjerklov and Saprykina, 08!

$\Lambda_b = 0.5423122$
$A = 1.015$
$B = 0.5020 \approx 0.5$
$\bar{A} = -0.7409$
$\bar{B} = 1.00035 \approx 1$

The bundle merging bifurcation

SNAs in the projectivized dynamics

The bundle merging bifurcation is a mechanism that produces SNAs in the (projectivized) dynamics of quasiperiodic linear skew products.

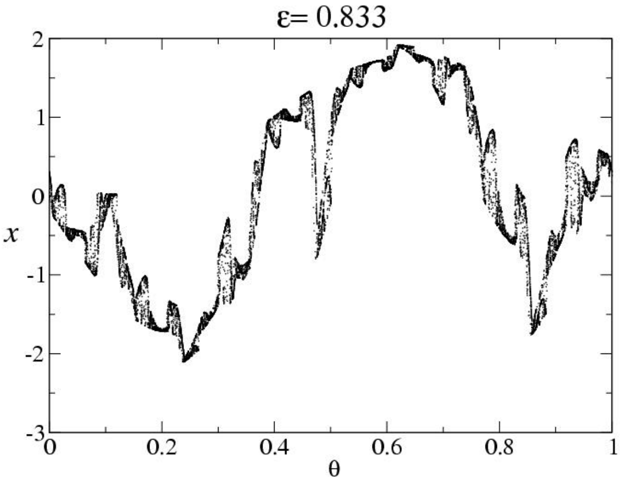
It is related with the transition from uniform hyperbolicity to non-uniform hyperbolicity.

For Harper maps, the projectivizations of Harper linear skew products, there are rigorous results: [Haro,Puig 06], [Jorba, Núñez, Obaya,Tatjer 07]

There are deep relations with the spectral properties of almost Mathieu operators.

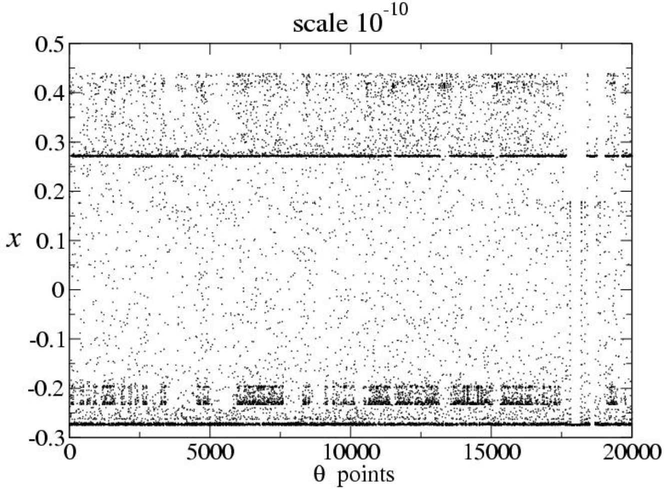
The fractalization route

Is this a SNA?



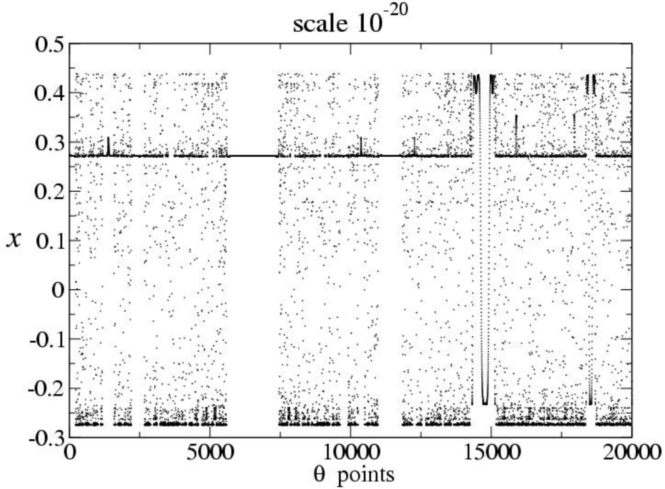
The fractalization route

Zooming



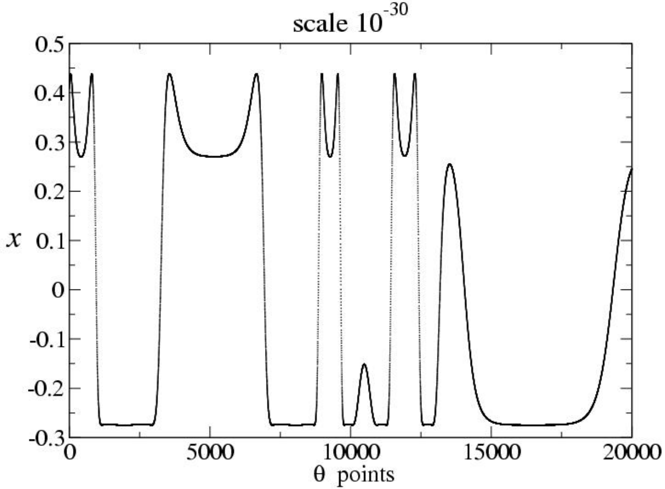
The fractalization route

Zooming again



The fractalization route

It is a regular curve!



Conclusions

- The formation of a strange non chaotic attractor for the linearized dynamics of an attracting torus produces a sudden growth of the spectrum (breakdown of exponential dichotomies and loss of reducibility).
- There are quantitative regularities for Λ and Δ, and some of them have been proved in specific models.
- This seems to be the prelude of the destruction of the torus when the upper Lyapunov multiplier (the external radius of the spectrum) crosses 1.
- Does it produces a formation of a strange chaotic attractor?

The phenomenon also happens for $d \neq 0$, when continuing an attracting invariant torus with fixed frequency (and adjusting parameters). [Canadell, Haro]

The rotating standard map

$$\begin{cases} \bar{\theta} = \theta + \omega \quad \pmod 1 \\ \bar{x} = x + \bar{y} \quad \pmod 1 \\ \bar{y} = y - \dfrac{\sin(2\pi x)}{2\pi}(K + \varepsilon \cos(2\pi\theta)) \end{cases}$$

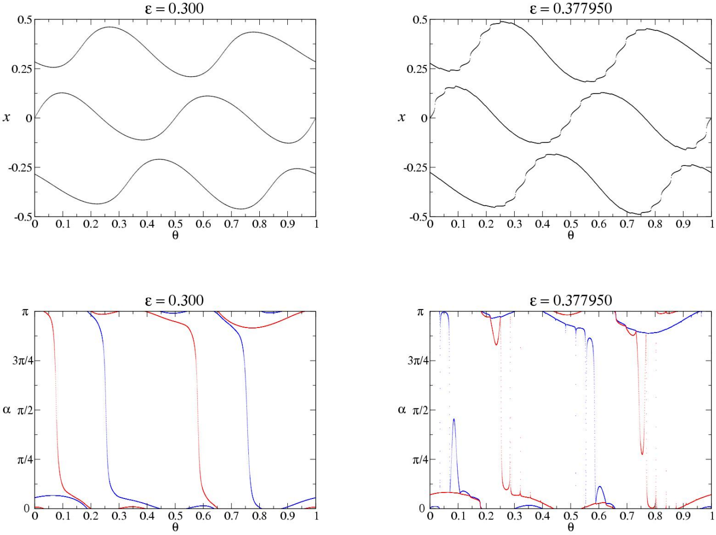
- K is the parameter of the standard map ($K = 0.2$);
- ε is the quasi-periodic parameter;
- ω is an algebraic number of order 3:

$$\omega = \sqrt[3]{\frac{19}{27} + \sqrt{\frac{11}{27}}} + \sqrt[3]{\frac{19}{27} - \sqrt{\frac{11}{27}}} - \frac{2}{3}\,.$$

[Artuso et al 91, Tompaidis 96, Haro 98]

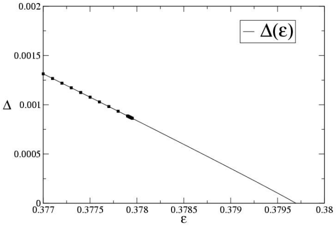
Bundle merging causing breakdown

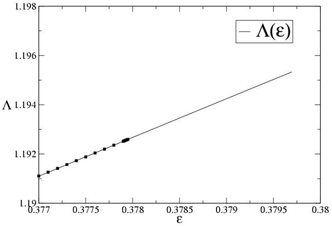
A 3-periodic torus close to breakdown, and projectivized bundles



Bundle merging causing breakdown

Quantitative estimates (universal laws)





$$\Delta_\varepsilon \sim \alpha(\varepsilon_c - \varepsilon)^\beta \text{ if } \varepsilon \leq \varepsilon_c$$

$$\Lambda_\varepsilon \sim \Lambda_c + A(\varepsilon_c - \varepsilon)^B \text{ if } \varepsilon \leq \varepsilon_c$$

$\varepsilon_c = 0.3796965$
$\alpha = 0.4063$
$\beta = 0.9693 \approx 1$

$\Lambda_c = 1.19533$
$A = -1.6$
$B = 1.00 \approx 1$

Conclusions

- The formation of a SNA in the linearized dynamics of a saddle type torus produces the sudden growth of the spectrum.
- Since at the collapse, 1 is inside the spectrum, the torus is not normally hyperbolic and it breaks down.
- There are some conjectured regularities in the behavior of the observables Δ and Λ, but no proofs!

A 3D Fattened Arnold Family

The equations of the model

$$\begin{cases} \bar{x} = x + a + \varepsilon(\sin(x) + y + z/2) \pmod{2\pi} \\ \bar{y} = b(\sin(x) + y) \\ \bar{z} = c(\sin(x) + y + z) \end{cases}$$

where

- b, $c \in \mathbb{R}$ fixed parameters,
- $a \in \mathbb{R}^1$ adjusting parameter,
- $\varepsilon \in \mathbb{R}$ perturbation parameter.

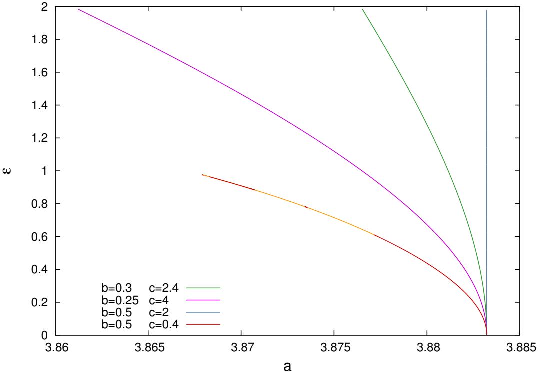
Continuation of NHIT: [Broer, Osinga, Vegter, 97] [Broer, Hagen, Vegter 07][Canadell, Haro 14]

Here: Continuation of **quasi-periodic NHIT** with frequency $\omega = \left(\sqrt{5} + 1\right)\pi$ w.r.t. ε. [Canadell, Haro 14]

A 3D Fattened Arnold Family

Continuations with respect to ε

DISSIPATIVE CASE	CONSERVATIVE CASE	REVERSIBLE CASE	ATTRACTING CASE
b = 0.3	b = 0.25	b = 0.5	b = 0.5
c = 2.4	c = 4	c = 2	c = 0.4

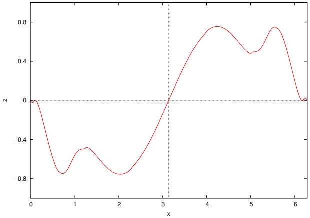


REVERSIBLE CASE

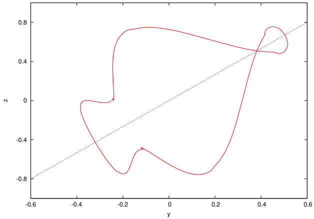
$b = 0.5, c = 2$

$$F_{a,\varepsilon} = I_1 I_0 \quad \text{for involutions:}$$

$$I_0\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -x + 2\pi \\ \sin(x) + y + \frac{3}{4}z \\ -z \end{pmatrix}, \quad I_1\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -x + \epsilon y + \frac{\epsilon}{4}z + a + 2\pi \\ \frac{y}{2} + \frac{3}{8}z \\ 2y - \frac{z}{2} \end{pmatrix}$$



(a)

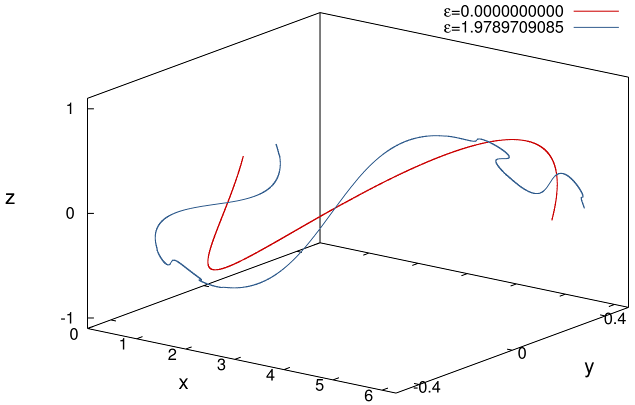


(b)

- Case (a) : 0th-symmetry $\Gamma_1 = \left\{(x, y, z) \in \mathbb{R}^3 \mid (\pi, y, 0)\right\}$.
- Case (b) : 1st-symmetry $\Gamma_0 = \left\{(x, y, z) \in \mathbb{R}^3 \mid (\frac{a}{2} + \pi + \frac{4}{3}\varepsilon z, \frac{3}{4}z, z)\right\}$.

REVERSIBLE CASE

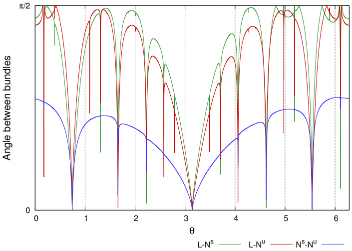
The first and the last tori

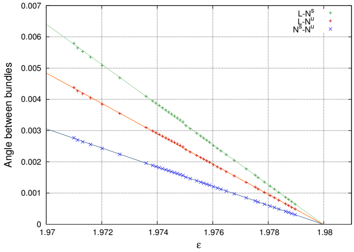


REVERSIBLE CASE

Breakdown due to triple collision of bundles!

ε	a	Λ_S	Λ_U	$\alpha(L, N^S)$	$\alpha(L, N^U)$	$\alpha(N^S, N^U)$
0.0000000000	3.8832220775	0.5000000000	2.0000000000	1.35380e+00	1.12244e+00	6.43501e-01
0.8000000000	3.8832220775	0.4960293703	2.0160096556	8.53835e-01	6.46182e-01	4.21770e-01
1.6000000000	3.8832220775	0.4732854745	2.1128896911	2.57675e-01	1.93476e-01	1.24632e-01
1.9000000000	3.8832220775	0.4448527617	2.2479347914	5.16573e-02	3.90081e-02	2.47141e-02
1.9789709085	3.8832220775	0.4312371565	2.3189096415	6.42944e-04	4.86248e-04	3.06950e-04





$$\alpha_{L,S} \simeq 1.27075384 - 0.64180240\ \varepsilon \Longrightarrow \varepsilon_{c_{LS}} \simeq 1.9799767600$$

$$\alpha_{L,U} \simeq 0.96088637 - 0.48530176\ \varepsilon \Longrightarrow \varepsilon_{c_{LU}} \simeq 1.9799771064$$ 1.97998

$$\alpha_{S,U} \simeq 0.60684869 - 0.30649291\ \varepsilon \Longrightarrow \varepsilon_{c_{SU}} \simeq 1.9799762457$$ 1.97998

REVERSIBLE CASE

Breakdown due to triple collision of bundles!

ε	a	Λ_S	Λ_U	$\alpha(L, N^S)$	$\alpha(L, N^U)$	$\alpha(N^S, N^U)$
0.0000000000	3.8832220775	0.5000000000	2.0000000000	1.35380e+00	1.12244e+00	6.43501e-01
0.8000000000	3.8832220775	0.4960293703	2.0160096556	8.53835e-01	6.46182e-01	4.21770e-01
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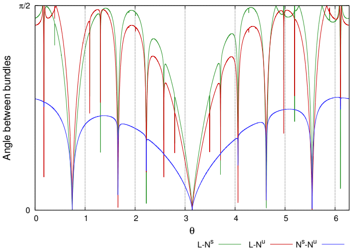
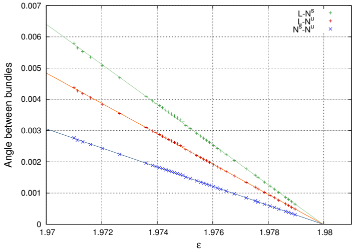
1.9000000000	3.8832220775	0.4448527617	2.2479347914	5.16573e-02	3.90081e-02	2.47141e-02
1.9789709085	3.8832220775	0.4312371565	2.3189096415	6.42944e-04	4.86248e-04	3.06950e-04



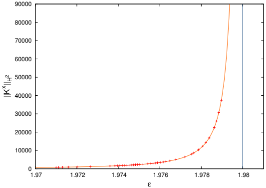


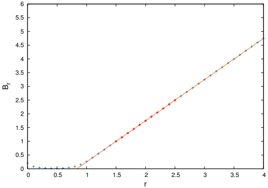
$$\alpha_{L,S} \simeq 1.27075384 - 0.64180240\ \varepsilon \Longrightarrow \varepsilon_{c_{LS}} \simeq 1.97998$$
$$\alpha_{L,U} \simeq 0.96088637 - 0.48530176\ \varepsilon \Longrightarrow \varepsilon_{c_{LU}} \simeq 1.97998$$
$$\alpha_{S,U} \simeq 0.60684869 - 0.30649291\ \varepsilon \Longrightarrow \varepsilon_{c_{SU}} \simeq 1.97998$$

REVERSIBLE CASE

Blow up of Sobolev norms

ε	a	H^2	C^1	$\mathrm{N_F}$
0.0000000000	3.8832220775	0.00000e+00	0.00000e+00	64
0.8000000000	3.8832220775	2.81522e−01	3.33102e−01	64
1.6000000000	3.8832220775	2.86960e+00	2.18468e+00	128
1.9000000000	3.8832220775	4.20873e+01	1.28306e+01	1024
1.9789709085	3.8832220775	3.77268e+04	4.69412e+02	1048576





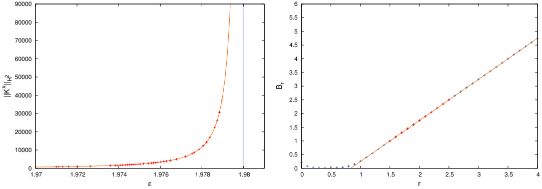
$$H^2(\varepsilon) \simeq \frac{0.21964940}{(1.97998184 - \varepsilon)^{1.74772558}} \implies \varepsilon_{c_{H^2}} \simeq 1.97998184$$

$$H^r(\varepsilon) \simeq \frac{A_r}{(\varepsilon_c - \varepsilon)^{B_r}}, B_r \simeq -1.252001 + 1.499865\ r \implies r_c \simeq 0.834743$$

REVERSIBLE CASE

Blow up of Sobolev norms

ε	a	H^2	C^1	N_F
0.0000000000	3.8832220775	0.00000e+00	0.00000e+00	64
0.8000000000	3.8832220775	2.81522e−01	3.33102e−01	64
1.6000000000	3.8832220775	2.86960e+00	2.18468e+00	128
1.9000000000	3.8832220775	4.20873e+01	1.28306e+01	1024
1.9789709085	3.8832220775	3.77268e+04	4.69412e+02	1048576



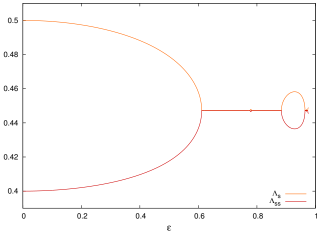
$$H^2(\varepsilon) \simeq \frac{0.21964940}{(1.97998184 - \varepsilon)^{1.74772558}} \quad \Longrightarrow \quad \varepsilon_{c_{H^2}} \simeq 1.97998$$

$$H^r(\varepsilon) \simeq \frac{A_r}{(\varepsilon_c - \varepsilon)^{B_r}}, B_r \simeq -1.252001 + 1.499865\ r \quad \Longrightarrow \quad r_c \simeq 0.834743$$

ATTRACTING CASE: NODE-FOCUS TORI

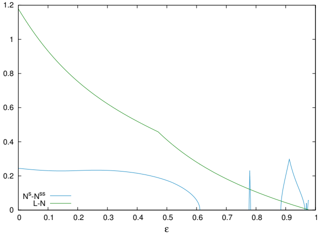
⇝ Continuation of an attracting QP-NHIT tori with a two dimensional stable bundle

ε	a	Λ_S	Λ_{SS}	$\alpha_{S,SS}$
0.00000000	3.88322208	0.50000000	0.40000000	2.4498e-01
0.20000000	3.88251749	0.49740876	0.40208379	2.3143e-01
0.40000000	3.88051059	0.48820774	0.40966167	2.1581e-01
0.60000000	3.87735825	0.45788972	0.43678639	5.8341e-02
0.61129562	3.87714576	0.44722008	0.44720711	3.6775e-05

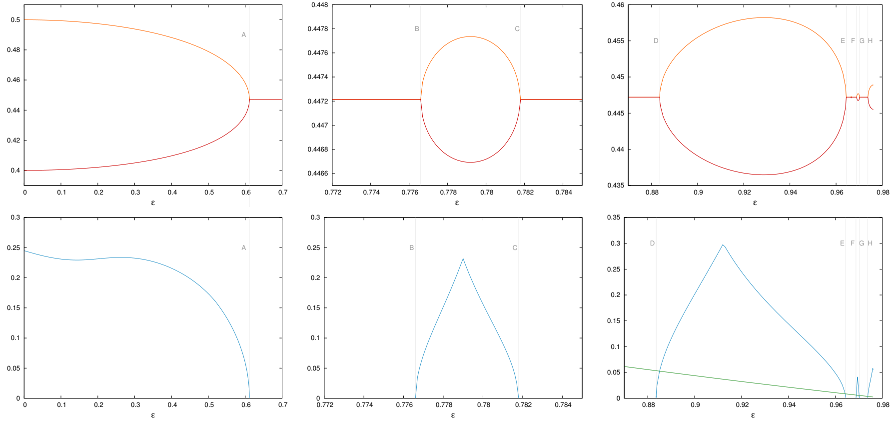


⇝ Node-Focus transition at $\varepsilon_{\rm A} \simeq 0.6112956166$ ⇝ collision of the eigenvalues + smooth collision between slow and fast subbundles ⇝ the torus persists

⇝ Several node-focus transitions after $\varepsilon_{\rm A}$

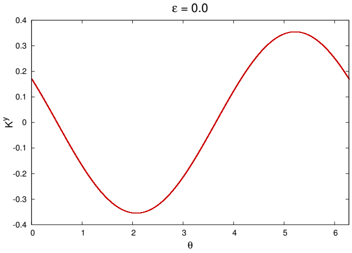


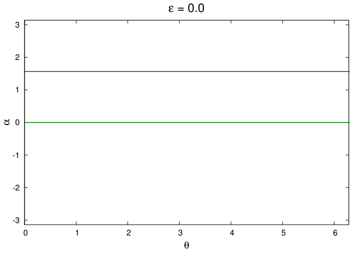
ATTRACTING CASE: NODE-FOCUS TORI



ATTRACTING CASE: NODE-FOCUS TORI

ε	a	Λ_L	λ_{n_1}	λ_{n_2}	index×2
0.0000000000	3.8832220775	1.00	0.5000000000	0.4000000000	0
0.6000000000	3.8773582471	1.00	0.4579030644	0.4367722458	0
0.7000000000	3.8753378014	1.00	$\pm$0.4472137102	$\pm$0.4472134760	-
0.7800000000	3.8734835655	1.00	0.4477095796	0.4467181597	13
0.8700000000	3.8711212578	1.00	$\pm$0.4472137080	$\pm$0.4472134930	-
0.9400000000	3.8690546139	1.00	$-$0.4577666206	$-$0.4369038657	-8
0.9650000000	3.8682620624	1.00	$\pm$0.4472138320	$\pm$0.4472133709	-
0.9665195313	3.8682129039	1.00	0.4473017886	0.4471254316	-152
0.9680000000	3.8681648979	1.00	$\pm$0.4472137160	$\pm$0.4472134867	-
0.9698878174	3.8681035237	1.00	0.4476003264	0.4468272104	81
0.9702000000	3.8680933571	1.00	$\pm$0.4472138298	$\pm$0.4472133728	-
0.9748679777	3.8679407539	1.00	$-$0.4486032989	$-$0.4458282085	-63
0.9761123718	3.8678998859	1.00	$-$0.4489059727	$-$0.4455276098	-63





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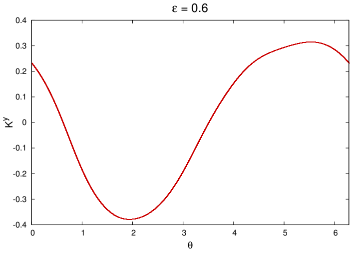

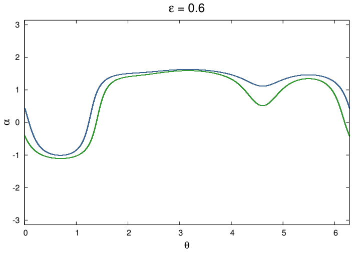

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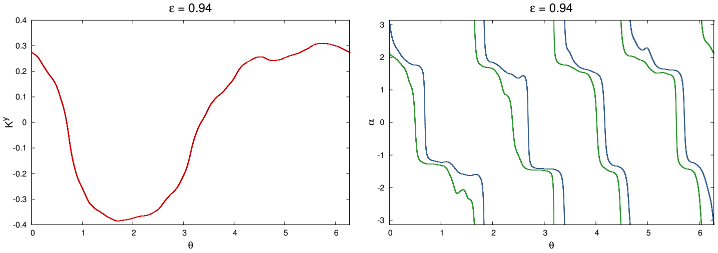

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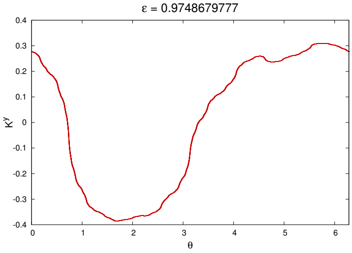
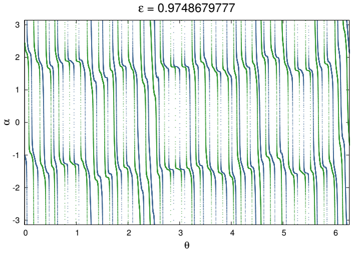
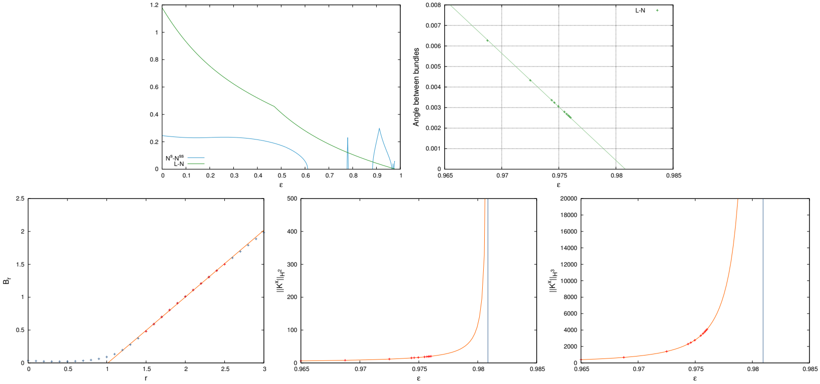




ATTRACTING CASE: NODE-FOCUS TORI



$$\alpha_{L,S} \simeq 1.270754 - 0.641802\ \varepsilon \quad \Longrightarrow \quad \varepsilon_{c_{LS}} \simeq 0.980821$$

$$H^2(\varepsilon) \simeq \frac{0.093014}{(0.980863 - \varepsilon)^{1.010053}} \quad \Longrightarrow \quad \varepsilon_{c_{H^2}} \simeq 0.980863$$

$$H^3(\varepsilon) \simeq \frac{0.108759}{(0.980941 - \varepsilon)^{1.981441}} \quad \Longrightarrow \quad \varepsilon_{c_{H^3}} \simeq 0.980941$$

$$B_r \simeq -1.031283 + 1.016700\ r \quad \Longrightarrow \quad r_c \simeq 1.014343$$

Bundle collisions

Findings in different contexts

	Continuation of qp NHIT	**Breakdown and bundle merging**
On skew-products over rotations [Haro, de la Llave 06,07] [Figueras, Haro 12]	$\rightsquigarrow$ attracting tori	Collision of slow and fast stable bundles prior to the breakdown
	$\rightsquigarrow$ saddle tori	Collision of stable and unstable bundles producing the breakdown
On conformally symplectic systems [Calleja, Figueras 12]	$\rightsquigarrow$ attracting tori	Collision of stable and tangent bundles producing the breakdown
On general systems [Canadell, Haro 14]	$\rightsquigarrow$ attracting tori	Sequence of node-focus transitions before collision of 2D stable and 1D tangent bundles
	$\rightsquigarrow$ saddle tori	Collision of tangent, stable and unstable bundles producing the breakdown

Some papers ...

• Àlex Haro, Rafael de la Llave, *Manifolds at the verge of a hyperbolicity breakdown*. Communication at the SIAM Conference on Applications of Dynamical Systems, 22-26 may 2005

• Àlex Haro, Rafael de la Llave, *Manifolds on the verge of a hyperbolicity breakdown.* (Chaos, 2006)

• Àlex Haro, Rafael de la Llave, *A parameterization method for the computation of invariant tori and their whiskers in quasi periodic maps: explorations and mechanisms for the breakdown of hyperbolicity.* (SIADS, 2007)

• Jordi Lluís Figueras, *Fiberwise Hyperbolic Invariant Tori in quasiperiodically skew product systems*, (PhD thesis, UB, 2011)

• Jordi Lluís Figueras, Àlex Haro, *Hyperbolicity breakdown in quasiperiodic area preserving maps*, preprint.

• Marta Canadell, *Computation of normally hyperbolic invariant manifolds.* (PhD thesis, UB, 2014)

• Marta Canadell, Àlex Haro, *Parameterization method for computing quasi-periodic reducible normally hyperbolic invariant tori*, (Advances in Differential Equations and Applications, 2014).

... and some movies

by Pedro Almodóvar