On dynamical properties of area-preserving maps with quadratic and cubic homoclinic tangencies

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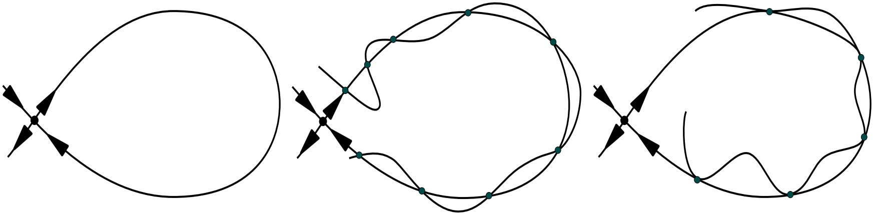
Introduction

- We study homoclinic orbits to saddle fixed points in area-preserving maps
- These orbits are of great interest, since they imply complicated dynamics



a) homoclinic connection (separatrix) b) transverse homoclinic orbit c) nontransversal homoclinic orbit (homoclinic tangency)

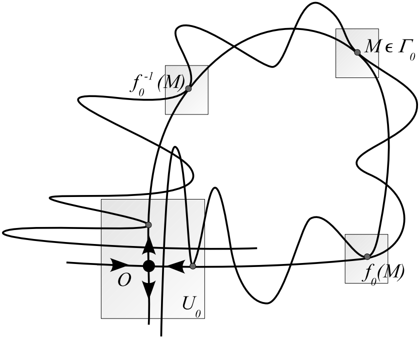
Bifurcations of homoclinic tangencies in area-preserving maps (APMs)

- 2D symplectic maps with quadratic homoclinic tangencies (conservative Hénon maps appear)
- Non-orientable APMs with quadratic homoclinic tangencies (conservative non-orientable Hénon maps appear)
- APMs with cubic homoclinic tangencies (conservative cubic Hénon maps appear)

Bifurcations of homoclinic tangencies in APMs

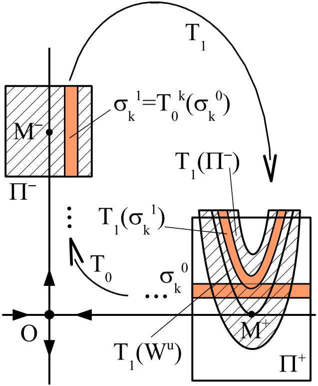
Goal:

To study the behavior of orbits near a nontransversal homoclinic trajectory to a saddle fixed point in APMs.



Question: What happens to periodic orbits near the homoclinic orbit?

- We consider a family f_ε with $f_\varepsilon\big|_{\varepsilon=0} \equiv f_0$
- We study bifurcations of single-round periodic orbits, i.e. periodic orbits which entirely lie in a neighborhood of the homoclinic orbit and pass close to it only once
- Method: to construct first return maps $T_k = T_0^k T_1$



- *local map*:
  $T_0^k(\varepsilon) : \Pi^+ \to \Pi^-$
- *global map*:
  $T_1(\varepsilon) : \Pi^- \to \Pi^+$

The fixed points of T_k $\Rightarrow$ single-round periodic orbits of f_ε

Homoclinic tangencies: Background

Dissipative case: Gavrilov and Shilnikov (1972, 1973)

- Bifurcations of single-round periodic orbits in diffeomorphisms with quadratic homoclinic tangencies
- Theorem on cascade of periodic sinks (sources)
  - homoclinic tangencies lead to the appearance of asymptotically stable (if $\sigma < 1$) or completely unstable (if $\sigma > 1$) periodic orbits
  - such orbits are observed at values of the splitting parameter μ belonging to an infinite sequence (cascade) of intervals
  - these intervals do not intersect and accumulate to $\mu = 0$

Homoclinic tangencies: Background

Conservative case:

- Newhouse (1977): homoclinic tangencies lead to the appearance of elliptic periodic orbits
- Biragov and Shilnikov (1989): the appearance of generic elliptic periodic orbits under bifurcations of a homoclinic loop of a saddle-focus
- Mora-Romero (1997): the appearance of generic elliptic periodic orbits under bifurcations of 2D symplectic maps with quadratic tangencies

But bifurcation diagrams were not done, since the coexistence of elliptic periodic orbits of arbitrarily large periods was not considered.

Bifurcations of homoclinic tangencies in area-preserving maps (APMs)

- Bifurcations of quadratic homoclinic tangencies for 2D symplectic maps
- Dynamics and bifurcations of non-orientable APMs with quadratic homoclinic tangencies
- Bifurcations of cubic homoclinic tangencies in APMs

Bifurcations of quadratic homoclinic tangencies for 2D symplectic maps

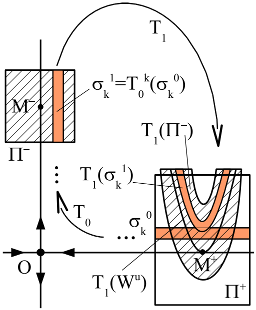
We consider a C^r-smooth ($r \geq 3$) symplectic map f_0 satisfying the conditions

- f_0 has a saddle fixed point O with multipliers λ and λ^{-1} ($|\lambda| < 1$)
- f_0 has a homoclinic orbit Γ_0 at whose points $W^s(O)$ and $W^u(O)$ have a quadratic tangency

We consider a family f_μ of symplectic maps close to f_0, where μ is the splitting parameter of the homoclinic tangency.

First return maps

We construct first return maps $T_k = T_0^k T_1$.



- $T_0^k(\varepsilon) : (x_0, y_0) \to (x_k, y_k)$
- $T_1(\varepsilon) : (x_k, y_k) \to (\bar{x}, \bar{y})$

Local map T_0 is the saddle map

$$\bar{x} = \lambda x + h_1(x, y, \varepsilon)x \ , \ \bar{y} = \lambda^{-1} y + h_2(x, y, \varepsilon)y.$$

Lemma (nth-order normal form of T_0)

There is a canonical change of coordinates, of class C^r for $n = 1$ or C^{r-2n} for $n \geq 2$, that brings T_0 to the form

$$\bar{x} = \lambda x \left(1 + \beta_1 \cdot xy + \cdots + \beta_n \cdot (xy)^n\right) + h.o.t.$$
$$\bar{y} = \lambda^{-1} y \left(1 + \hat{\beta}_1 \cdot xy + \cdots + \hat{\beta}_n \cdot (xy)^n\right) + h.o.t$$

$T_0^k : (x_0, y_0) \to (x_k, y_k)$

Lemma (kth iteration of T_0)

The map T_0^k is written, for any integer k, as follows

$$
\begin{aligned}
x_k &= \lambda^k x_0 \cdot R_n^{(k)}(x_0 y_k, \varepsilon) + \lambda^{(n+1)k} P_n^{(k)}(x_0, y_k, \varepsilon), \\
y_0 &= \lambda^k y_k \cdot R_n^{(k)}(x_0 y_k, \varepsilon) + \lambda^{(n+1)k} Q_n^{(k)}(x_0, y_k, \varepsilon),
\end{aligned}
$$

where

- $R_n^{(k)} \equiv 1 + \tilde{\beta}_1(k)\lambda^k x_0 y_k + \cdots + \tilde{\beta}_n(k)\lambda^{nk}(x_0 y_k)^n$,
- $\tilde{\beta}_i(k)$ *are polynomials of degree* i
- *the functions* $P_n^{(k)}, Q_n^{(k)} = o\left(x_0^n y_k^n\right)$ *are uniformly bounded in* k

Global map T_1

$$\bar{x} - x^+ = F(x, y - y^-, \varepsilon), \;\; \bar{y} = G(x, y - y^-, \varepsilon),$$

where

- $F(0) = G(0) = 0$, $G_y(0) = 0, G_{yy}(0) = 2d \neq 0$.
- $$\begin{array}{rcl} F(x, y - y^-, \varepsilon) & = & ax + b(y - y^-) + e_{20}x^2 + e_{11}x(y - y^-) \\ & & + e_{02}(y - y^-)^2 + h.o.t. \\ G(x, y - y^-, \varepsilon) & = & \mu + cx + d(y - y^-)^2 + f_{20}x^2 + f_{11}x(y - y^-) \\ & & + f_{30}x^3 + f_{21}x^2(y - y^-) + f_{12}x(y - y^-)^2 \\ & & + f_{03}(y - y^-)^3 + h.o.t. \, , \end{array}$$
- $\mu = \varepsilon_1 \equiv G(0, 0, \varepsilon)$ is the splitting parameter
- $J(T_1) = -bc \equiv 1$, since T_1 is symplectic

Lemma

For every sufficiently large k the map T_k can be brought to the form

$$\bar{X} = Y + k\lambda^{2k}\varepsilon_k^1,$$
$$\bar{Y} = M - X - Y^2 + \frac{f_{03}}{d^2}\lambda^k Y^3 + k\lambda^{2k}\varepsilon_k^2\ ,$$

- $\varepsilon_k^{1,2}(X, Y, M)$ *are uniformly bounded in* k
- $M = -d(1 + \nu_k^1)\lambda^{-2k}(\mu + \lambda^k y^-(sign(c)|\lambda|^{-\tau} - 1)(1 + k\beta_1\lambda^k x^+ y^-)) - s_0 + \nu_k^2$
- $s_0 = dx^+(ac + f_{20}x^+) - f_{11}x^+(1 - \frac{1}{4}f_{11}x^+)$
- $\nu_k^1 = O(\lambda^k)$, $\nu_k^2 = O(k\lambda^k)$
- $\tau = -\frac{1}{\ln|\lambda|}\ln\left|\frac{cx^+}{y^-}\right|$

Limit form of T_k is the conservative Hénon map*

$$\bar{x} = y,\ \bar{y} = M - x - y^2.$$

- $M \in (-1; 3)$: generic elliptic fixed point (except for $M = 0$ and $M = 5/4$)
- $M = 5/4$: non-degenerate $1:3$ resonance
- $M = 0$: degenerate $1:4$ resonance
- $M = -1$: parabolic fixed point with multipliers $\nu_1 = \nu_2 = +1$
- $M = 3$: parabolic fixed point with multipliers $\nu_1 = \nu_2 = -1$

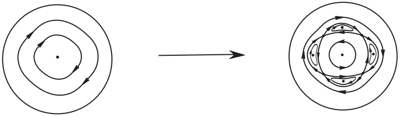
*Recall that the Hénon map [Hénon; 1976] is $\bar{x} = y,\ \bar{y} = 1 - bx + ay^2$. In the coordinates $x_{new} = -ax, y_{new} = -ay$, the map is written in the form $\bar{x} = y,\ \bar{y} = M_1 - M_2 x - y^2$ with $M_1 = -a$ and $M_2 = b$. For $M_2 = b = 1$ it is conservative.

In the *refined* map

$$\bar{x} = y, \; \bar{y} = M - x - y^2 + \frac{f_{03}}{d^2}\lambda^k y^3,$$

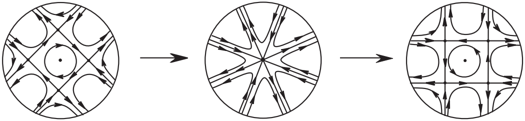
with $f_{03} \neq 0$, the resonance 1 : 4 becomes non-degenerate [Biragov; 87], [MGonchenko; 2005].

$f_{03}\lambda^k \; > \; 0$:



$\beta \leq 0$ $\qquad$ $\beta > 0$

$f_{03}\lambda^k < 0$:



$\beta < 0$ $\qquad$ $\beta = 0$ $\qquad$ $\beta > 0$

- We express μ in terms of M (see equations for M in the Rescaling Lemma)
- $M \in (-1; 3)$ is translated into intervals δ_k of parameter μ with boundaries $\mu_k^\pm$ for f_μ:

$$\mu_k^+ = -\lambda^k y^-(sign(c)|\lambda|^{-\tau} - 1)(1 + k\beta_1\lambda^k x^+ y^-) - \frac{1}{d}(-1 + s_0 + \hat{\rho}_k)\lambda^{2k}$$

$$\mu_k^- = -\lambda^k y^-(sign(c)|\lambda|^{-\tau} - 1)(1 + k\beta_1\lambda^k x^+ y^-) - \frac{1}{d}(3 + s_0 + \hat{\rho}_k)\lambda^{2k}$$

Thus, we get bifurcation scenarios for single-round periodic orbits in one parameter family f_μ

Main result I (symplectic case)

Theorem (On one parameter cascade of elliptic points for f_μ)

1. *In any segment $[-\mu_0, \mu_0]$ of values of μ, $\exists$ infinitely many open intervals δ_k, for $k = \bar{k}, \bar{k}+1, \dots,$ such that the map f_μ has a single-round periodic elliptic orbit at $\mu \in \delta_k$;*
2. *At the border points $\mu = \mu_k^+$ and $\mu = \mu_k^-$ of δ_k, f_μ has a single-round parabolic periodic orbit with double multipliers $+1$ or -1, respectively;*
3. *The elliptic orbit is generic (KAM-stable) for $\mu \in \delta_k$, except for two values corresponding to the $1:3$ and $1:4$ resonances;*
4. *In the cases $c < 0$ and $c > 0$ with $\tau \neq 0$, the intervals δ_i and δ_j do not intersect for $i \neq j$.*

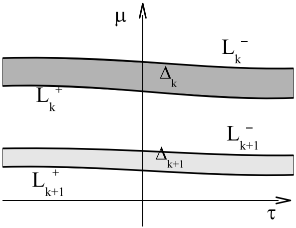
Item 4 is new and important in describing of bifurcation diagrams.

Homoclinic invariant of f_0

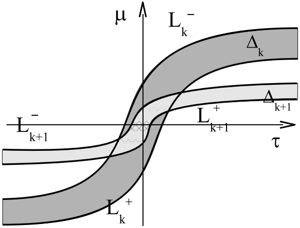
- τ is a homoclinic invariant of f_0 responsible for the presence of the chaotic dynamics:
  - $\tau > 0$: f_0 has infinitely many Smale horseshoes
  - $\tau < 0$: dynamics of f_0 is trivial
- At $\tau = 0$ and some values of s_0, all intervals δ_k (for $k = \bar{k}, \bar{k} + 1, \ldots$) contain $\mu = 0$, i.e. the map f_0 has infinitely many coexisting elliptic periodic points of all successive periods $k = \bar{k}, \bar{k} + 1, \ldots$ (*global resonance*)
- The case $\tau = 0, \mu = 0$ requires to consider two parameter family $f_{\mu,\tau}$

Theorem (On two parameter cascades of elliptic points for $f_{\mu,\tau}$)

1. *In any neighborhood of the origin in the* (τ, μ)*-plane,* $\exists$ *infinitely many open domains* Δ_k*, for* $k = \bar{k}, \bar{k}+1, \dots,$ *such that* $f_{\mu,\tau}$ *has a single-round periodic elliptic orbit in* Δ_k*;*
2. *The domains* Δ_k *accumulate to the axis* $\mu = 0$ *as* $k \to \infty$*;*
3. *The boundaries of* Δ_k *are two curves* L_k^+ *and* L_k^- *where* $f_{\mu,\tau}$ *has a parabolic single-round orbit with double multipliers* $+1$ *at and* -1*;*
4. *The elliptic orbit is generic (KAM-stable) for* $(\tau, \mu) \in \Delta_k$*, except for* $(\tau, \mu) \in L_k^{\pi/2}$ *(1:4 resonance) and* $(\tau, \mu) \in L_k^{2\pi/3}$ *(1:3 resonance)*
5. *In the case* $c < 0$*,* Δ_i *and* Δ_j *do not intersect for* $i \neq j$
6. *In the case* $c > 0$*,* Δ_i *and* Δ_j *are crossed and they intersect the axis* $\mu = 0$*; Moreover, for* $-3 < s_0 < 1$*,* Δ_k *contains the origin* $(\tau = 0, \mu = 0)$*.*



Case I: $c<0$



Case II: $c>0$

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Non-orientable APMs

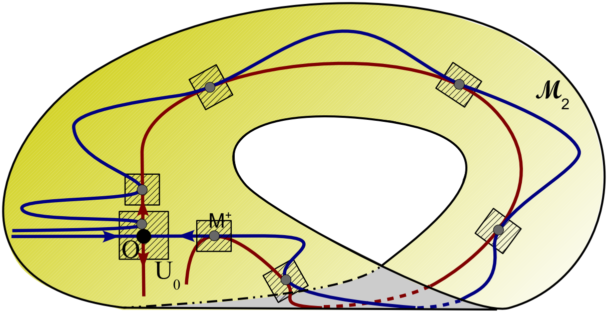
We consider a C^r-smooth ($r \geq 3$) non-orientable area-preserving map f_0 satisfying the conditions

- f_0 has a saddle fixed point O with multipliers λ and γ ($0 < |\lambda| < 1 < |\gamma|$) and $|\lambda\gamma| = 1$
- f_0 has a homoclinic orbit Γ_0 at whose points $W^s(O)$ and $W^u(O)$ have a quadratic tangency

We divide such APMs into 2 groups:

- the globally non-orientable maps with an orientable saddle ($\lambda\gamma = 1$) on a non-orientable surface (Möbius strip, Klein bottle, etc)
- the locally non-orientable maps with non-orientable saddles ($\lambda\gamma = -1$)

A globally non-orientable APM



First return maps

Limit form of T_k is the conservative Hénon map

$$\bar{x} = y, \; \bar{y} = M - \nu x - y^2.$$

- Globally non-orientable case: $\nu = -1$ (non-orientable conservative Hénon map)
- Locally non-orientable case: $\nu = 1$ for even k; $\nu = -1$ for odd k.

Non-orientable conservative Hénon map

$\nu = -1$:

$$\bar{x} = y,\ \bar{y} = M + x - y^2.$$

The map has NO elliptic fixed points!!

- $M < 0$: no fixed points
- $M = 0$: a fixed point with multipliers (1, -1)
- $M > 0$: two saddle fixed points
- $M \in (0, 1)$: elliptic 2-periodic point
- $M = 1$: the period-doubling bifurcation occurs

Main result I (non-orientable case)

Theorem (One parameter cascades of elliptic points in APMs)

For any interval $(-\mu_0, \mu_0)$, $\exists$ a positive integer $\bar{k}$ such that the following holds:

1. (a). *In the globally non-orientable case, f_μ has no single-round elliptic periodic orbits, while $\exists$ infinitely many intervals e_k^2, $k = \bar{k}, \bar{k}+1, \dots,$ where f_μ has a double-round elliptic orbit.*
   (b). *In the locally non-orientable case, $\exists$ infinitely many intervals e_{2m} and e_{2m+1}^2 such that f_μ has a single-round elliptic periodic orbit at $\mu \in e_{2m}$ and a double-round elliptic periodic orbit at $\mu \in e_{2m+1}^2$.*
2. *The intervals e_k as well as e_k^2 accumulate to $\mu = 0$ as $k \to \infty$ and do not intersect for sufficiently large and different integer k if $\alpha \neq 0$ and $\hat{\alpha} \neq 0$.*
3. *Interval e_k has border points $\mu = \mu_k^+$ and $\mu = \mu_k^-$ where the map f_μ has a single-round periodic orbit with multipliers $(1, 1)$ and with multiplier $(-1, -1)$. e_k^2 has border points $\mu = \mu_k^{2+}$ and $\mu = \mu_k^{2-}$ where the map f_μ has a single-round periodic orbit with multipliers $(1, -1)$ at and a double-round periodic orbit with multipliers $(-1, -1)$.*
4. *The elliptic orbit is generic (KAM-stable) in e_k and e_k^2, except for strong resonances 1:3 and 1:4.*

Homoclinic invariants

- The homoclinic invariants of f_0

$$\alpha = \frac{cx^+}{y^-} - 1 \text{ and } \hat{\alpha} = \frac{cx^+}{y^-} + 1$$

  are responsible for the presence of chaotic dynamics.
- The cases $\mu = 0, \alpha = 0$ and $\mu = 0, \hat{\alpha} = 0$ require to consider $f_{\mu,\alpha}$ and $f_{\mu,\hat{\alpha}}$.
- Globally non-orientable APMs: $f_{\mu,\alpha}$
- Locally non-orientable APMs: $f_{\mu,\alpha}$ if $c > 0$ and $f_{\mu,\hat{\alpha}}$ if $c < 0$.

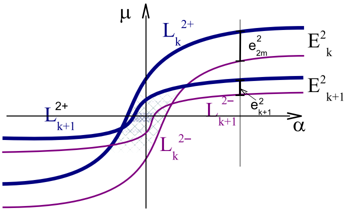
Main result II (non-orientable case)

Theorem (For two parameter families $f_{\mu,\alpha}$ and $f_{\mu,\hat{\alpha}}$)

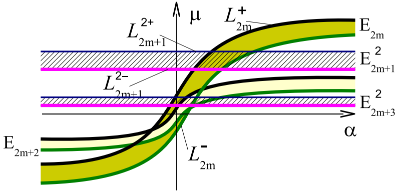
$\exists$ infinitely many open domains E_k^2 in the globally non-orientable case and domains E_{2m} and E_{2m+1}^2 in the locally non-orientable case, such that

1. *The map $f_{\mu,\alpha}$ and $f_{\mu,\hat{\alpha}}$ have a single-round periodic elliptic orbit in E_k, and have a double-round elliptic periodic orbit in E_k^2;*
2. *The domains E_k and E_k^2 accumulate to the axis $\mu = 0$ as $k \to \infty$;*
3. *Domain E_k has two boundaries, curves L_k^+ and L_k^-, corresponding to a single-round parabolic periodic orbit with multipliers $(1,1)0$ and $(-1,-1)$;*
4. *Domain E_k^2 has two boundaries, curves L_k^{2+} and L_k^{2-}, corresponding to a single-round parabolic periodic orbit with multipliers $(1,-1)$ and a double-round parabolic periodic orbit with multipliers $(-1,-1)$;*
5. *In the globally non-orientable case: the domains E_i^2 and E_j^2 with sufficiently large $i \neq j$ are crossed in the (μ,α)-plane and they intersect the axis $\mu = 0$.*
6. *In the locally non-orientable case: in the (μ,α)-plane, the domains E_{2i} and E_{2j} are crossed for sufficiently large $i \neq j$ and intersect all domains E_{2m+1}^2 as well as the axis $\mu = 0$, but the domains E_{2i+1}^2 and E_{2j+1}^2 do not intersect for $i \neq j$.*
   In the $(\mu,\hat{\alpha})$-plane, the domains E_{2i+1}^2 and E_{2j+1}^2 are crossed and they intersect all domains E_{2m} as well as the axis $\mu = 0$, but the domains E_{2i} and E_{2j} do not intersect for $i \neq j$.

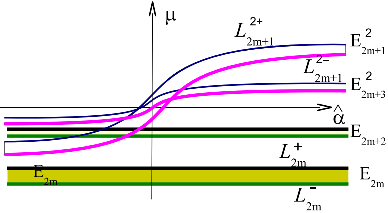
Domains E_k and E_k^2



(a) for globally non-orientable $f_{\mu,\alpha}$



(b) for locally non-orientable $f_{\mu,\alpha}$



(c) for locally non-orientable $f_{\mu,\hat{\alpha}}$

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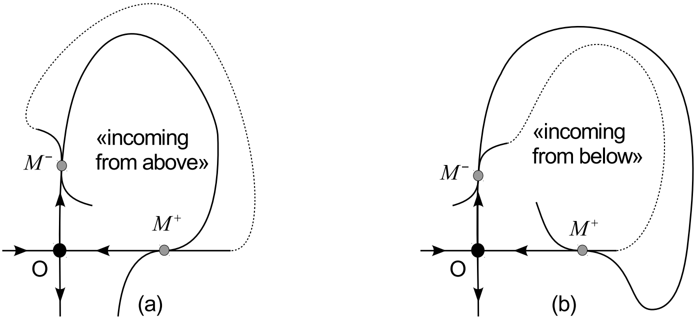
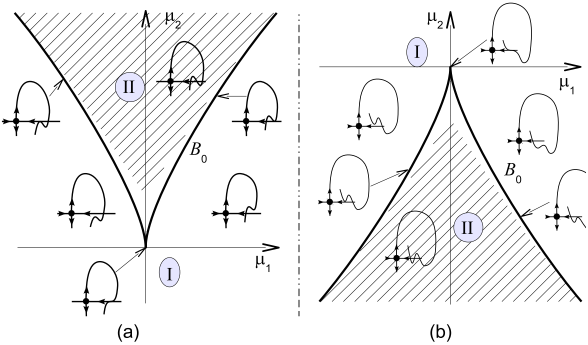
Cubic homoclinic tangencies

- We consider 2D symplectic maps f_0 with a homoclinic orbit Γ_0 along which the stable and unstable invariant manifolds have a cubic homoclinic tangency.
- We distinguish two types of cubic homoclinic tangencies: “incoming from above” and “incoming from below”.



Bifurcation curve B_0

- Bifurcations of cubic homoclinic tangencies are codim 2 bifurcations and, thus, we consider a family f_{μ_1,μ_2}.
- There is a curve B_0 where f_{μ_1,μ_2} has a quadratic homoclinic tangency with a transverse homoclinic orbit.



Lemma (Rescaling Lemma for cubic homoclinic tangencies)

For every sufficiently large k the first return map T_k can be brought, by a linear transformation of coordinates and parameters, to the following form

$$\begin{aligned} \bar{x} &= y + O(\lambda^k), \\ \bar{y} &= M_1 - x + M_2 y + \nu y^3 + O(\lambda^k), \end{aligned} \qquad (1)$$

where

$$\nu = \text{sign}\,(d\lambda^k), \qquad (2)$$

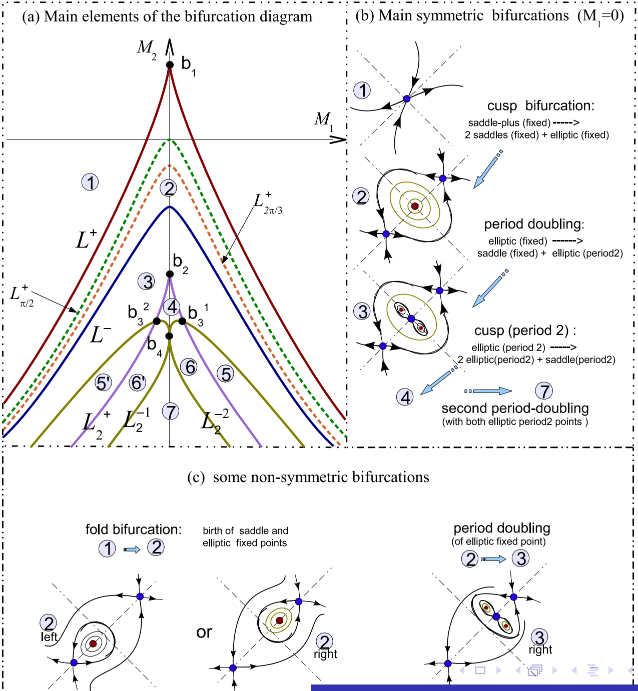
$$\begin{aligned} M_1 &= \sqrt{|d|}\lambda^{-3k/2}\left(\mu_1 - \lambda^k(y^- - cx^+) + O\left(k\lambda^{2k}\right)\right), \\ M_2 &= \lambda^{-k}\left(\mu_2 + f_{11}\lambda^k x^+ + O\left(k\lambda^{2k}\right)\right) \end{aligned} \qquad (3)$$

and $f_{11} = G_{xy}(0)$.

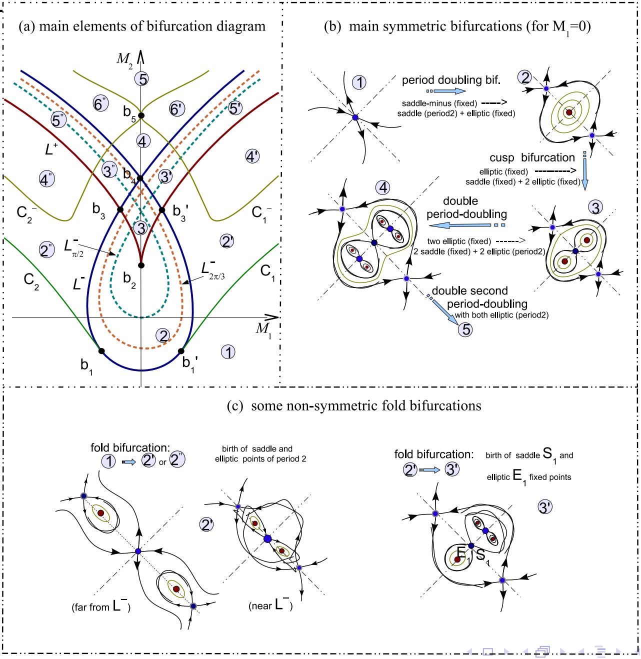
The limit form of T_k are conservative cubic Hénon maps

$$\bar{x} = y, \quad \bar{y} = M_1 - x + M_2 y + \nu y^3$$

$\nu = 1$:


(a) Main elements of the bifurcation diagram
M_2
b_1
M_1
1
2
$L^+_{2\pi/3}$
L^+
b_2
3
$L^+_{\pi/2}$
b_3^2
4
b_3^1
L^-
b_4
6
5
5'
6'
L_2^+
L_2^{-1}
7
L_2^{-2}
(b) Main symmetric bifurcations (M_1=0)
cusp bifurcation:
saddle-plus (fixed) ------>
2 saddles (fixed) + elliptic (fixed)
period doubling:
elliptic (fixed) ------>
saddle (fixed) + elliptic (period2)
cusp (period 2) :
elliptic (period 2) ------>
2 elliptic(period2) + saddle(period2)
second period-doubling
(with both elliptic period2 points)
(c) some non-symmetric bifurcations
fold bifurcation:
1 → 2
birth of saddle and
elliptic fixed points
period doubling
(of elliptic fixed point)
2 → 3
2 left
or
2 right
3 right

$\nu = -1$



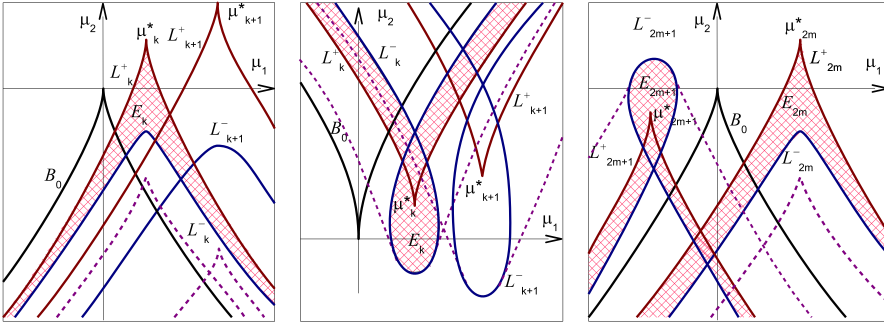
(a) main elements of bifurcation diagram
(b) main symmetric bifurcations (for M_1=0)
period doubling bif.
saddle-minus (fixed) -----> saddle (period2) + elliptic (fixed)
cusp bifurcation
elliptic (fixed) ---------> saddle (fixed) + 2 elliptic (fixed)
double period-doubling
two elliptic (fixed) ------> 2 saddle (fixed) + 2 elliptic (period2)
double second period-doubling
with both elliptic (period2)
(c) some non-symmetric fold bifurcations
fold bifurcation: 1 ⟹ 2' or 2"
birth of saddle and elliptic points of period 2
(far from L^-)
(near L^-)
fold bifurcation: 2' ⟹ 3'
birth of saddle S_1 and elliptic E_1 fixed points

Main result (cubic homoclinic tangencies)

Theorem (On the structure of the bifurcational diagram in f_{μ_1,μ_2})

1) *In any neighborhood of the origin in the* (μ_1,μ_2)*-plane, there exist infinitely many bifurcation curves* L_k^+ *and* L_k^- *as well as* $C_{1,2}^{k+}$ *and* $C_{1,2}^{k-}$ *that accumulate to the curve* B_0 *as* $k\to\infty$*. The map* f_{μ_1,μ_2} *has a parabolic single-round periodic orbit with multipliers* $\nu_1=\nu_2=+1$ *(respectively,* $\nu_1=\nu_2=-1$*) at* $\mu\in L_k^+$ *(respectively,* $\mu\in L_k^-$*), a double-round periodic orbit with multipliers* $\nu_1=\nu_2=+1$ *(respectively,* $\nu_1=\nu_2=-1$*) at* $\mu\in C_{1,2}^{k+}$ *(respectively,* $\mu\in C_{1,2}^{k-}$*).*

2) *For any sufficiently large* k*, in the* (μ_1,μ_2)*-plane there is a domain* E_k *between the curves* L_k^+ *and* L_k^- *where the map* f_{μ_1,μ_2} *has a single-round elliptic periodic orbit at* $\mu\in E_k$*. This point is generic (KAM-stable) for all* μ*, except for the ones corresponding to resonances* $1:3$ *and* $1:4$*.*



(a) d>0, λ>0 (b) d<0, λ>0 (c) d>0, λ<0

Thank you for your attention

and

Moltíssimes felicitats, Lluís!!