# Entropy bounds for orbit types

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If $f$ is a continuous map of an interval, the **type** of a finite (exactly) invariant set $f(S) = S = \{p_1 < p_2 < \ldots < p_n\}$ is the permutation $\theta$ defined by $f(p_i) = p_{\theta(i)}$.

We write $P_n$ for the permutations on $\{1, 2, \ldots, n\}$ and $C_n$ for the subset of cyclic ones, corresponding to single periodic orbits.

The **entropy** $h(\theta)$ of a type $\theta$ is the topological entropy it forces a continuous map to have:

$$h(\theta) := \inf\{h(f) | f \text{ has an invariant set of type } \theta\}.$$

Define

$$H(P_n) = \max_{\theta \in P_n} h(\theta), \quad H(C_n) = \max_{\theta \in C_n} h(\theta).$$

Question: What can be said about $H(P_n)$ and $H(C_n)$, and the permutations and cycles achieving the maximum?

Theorem

*(Misiurewicz and Nitecki 1991)*

$$\lim_{n\to\infty} \frac{1}{n} \exp H(P_n) = \lim_{n\to\infty} \frac{1}{n} \exp H(C_n) = 2/\pi.$$

In other words, the maximum entropy of $n$-permutations or of $n$-cycles grows like $\log(2n/\pi)$.

They gave explicit bounds for cycles, which serve also for permutations:

### Theorem

*(Misiurewicz-Nitecki 1991)*

$$\left(1-\frac{4}{n}\right)\left[\frac{2}{\pi}-\frac{1}{2\sqrt{n-4}}\right] \leq \frac{1}{n}\exp H(C_n) \leq \left(1+\frac{3}{n}\right)\left[\frac{2}{\pi}+\frac{1}{\sqrt{n}}\right].$$

The goal here is to sharpen these bounds, at least slightly, in light of subsequent developments.

For $n$ odd, define $l := \lfloor (n-1)/4 \rfloor$ so that $n = 4l+1$ or $n = 4l+3$. Define the (generalized) **Misiurewicz-Nitecki** orbit type $\theta_n$ of period $n$ by

$$j \mapsto \begin{cases} n-2l-j & \text{if} \quad 1 \le j < n-2l \quad \text{and } j \text{ odd;} \\ j-n+2l+1 & \text{if} \quad n-2l \le j \le n \quad \text{and } j \text{ odd;} \\ n-2l+j-1 & \text{if} \quad 1 \le j \le 2l \quad \text{and } j \text{ even;} \\ n+2l-j+2 & \text{if} \quad 2l < j \le n \quad \text{and } j \text{ even.} \end{cases}$$

For example, $\theta_5 = (1\ 2\ 4\ 5\ 3)$, $\theta_7 = (1\ 4\ 7\ 3\ 2\ 6\ 5)$, and $\theta_{11} = (1\ 6\ 11\ 5\ 2\ 8\ 9\ 3\ 4\ 10\ 7)$.

It is easy to check that $\theta_n$ is a cycle of period $n$.

Misiurewicz and Nitecki used these for $n = 4l+1$ in the proof of their lower bound.

In fact, they turn out to be entropy maximizers:

**Theorem**

*(G-Tolosa 1992) For $n$ odd, the Misiurewicz-Nitecki orbit types $\theta_n$ have maximum entropy among all $n$-permutations:*

$$h(\theta_n) = H(P_n) = H(C_n).$$

**Theorem**

*(G-Weiss 1993) For $n$ odd, the Misiurewicz-Nitecki orbit types $\theta_n$ are (essentially) uniquely maximal.*

There are analogues of these permutations in the even case, but they are not cyclic.

### Theorem

*(King 1997, G-Zhang 1998, King 2000) For $n$ even, there are essentially unique permutations $\theta_n$ (noncyclic for $n > 4$) having maximum entropy among all $n$-permutations:*

$$h(\theta_n) = H(P_n) > H(C_n).$$

The even cyclic case is harder, with the maximality part (minus uniqueness) of the following filling an entire AMS Memoir.

## Theorem

*(King-Strantzen 2001,2003) For $n = 4l$, there are essentially unique cyclic permutations $\psi_n$ having maximum entropy among all n-cycles:*

$$h(\psi_n) = H(C_n).$$

The remaining even case, $n = 4l + 2$, is still unresolved for cycles. However, there is a family of essentially unique cycles which are conjectured to achieve the maximum (Alseda-Juher-King, 2008).

For cycles, we have the following bounds:

### Theorem

*If $n$ is odd then*

$$\left(1-\frac{1}{n}\right)\left[\frac{2}{\pi}-\frac{1}{2\sqrt{n-1}}\right] \le \frac{1}{n}\exp H(C_n) \le \left(1-\frac{1}{n}\right)\left[\frac{2}{\pi}+\frac{1}{\sqrt{n-1}}\right].$$

*If $n$ is even then*

$$\left(1-\frac{2}{n}\right)\left[\frac{2}{\pi}-\frac{1}{2\sqrt{n-2}}\right] \le \frac{1}{n}\exp H(C_n) \le \frac{2}{\pi}+\frac{1}{\sqrt{n}}.$$

For permutations with $n$ odd, the bounds for $n$-cycles just stated apply. For permutations with $n$ even, the following sharper lower bounds hold:

### Theorem

*If $n = 4l$ is a multiple of 4 then*

$$\left(1 - \frac{1}{n}\right)\left[\frac{2}{\pi} - \frac{1}{2\sqrt{2}}\frac{\sqrt{n-2}}{n-1}\right] \leq \frac{1}{n}\exp H(P_n) \leq \frac{2}{\pi} + \frac{1}{\sqrt{n}}.$$

*If $n = 4l + 2$ then*

$$\left(1 - \frac{1}{n}\right)\left[\frac{2}{\pi} - \frac{1}{2}\frac{\sqrt{n-2}}{n-1}\right] \leq \frac{1}{n}\exp H(P_n) \leq \frac{2}{\pi} + \frac{1}{\sqrt{n}}.$$

Given a permutation $\theta \in P_n$, let $f_\theta$ be the piecewise linear interpolation of the graph of $\theta$ on the interval $[1, n]$.

The induced matrix $M(\theta)$ is the $(n-1) \times (n-1)$ matrix whose $(i, j)$th entry is 1 if $f_\theta([i, i+1]$ contains $[j, j+1]$ and 0 otherwise.

Then

$$h(\theta) = h(f_\theta) = \log \lambda,$$

where $\lambda = \rho(M(\theta))$ is the spectral radius of $M(\theta)$.

Induced matrices of permutations cannot have more than two 1's in the first or last column, four in the second or penultimate column, etc.

For example, the induced matrix of the Misiurewicz-Nitecki orbit type $\theta_7 = (1\ 4\ 7\ 3\ 2\ 6\ 5)$ is

$$M(\theta_7) = \begin{bmatrix} 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \end{bmatrix}$$

This is essentially a rotated, digitized graph of the piecewise linear function $f_{\theta_7}$.

Similarly, $\theta_{11} = (1\ 6\ 11\ 5\ 2\ 8\ 9\ 3\ 4\ 10\ 7)$ has induced matrix

$$M(\theta_{11}) = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \end{bmatrix}$$

To study the asymptotic shape and spectral radius of, for example, the $M(\theta_n)$, we embed the matrices into the bounded linear operators on $L^2[0,1]$, where their limit lives.

Embed $\mathbb{R}^m$ into $L^2[0,1]$ via step functions:

Let

$$J_m : \ \mathbb{R}^m \to L^2[0,1]$$

be given by $u = J_m(r)$, where $r = (r_1, \ldots, r_m) \in \mathbb{R}^m$ and $u(t) = r_i$ if $(i-1)/m \le t < i/m$.

So $u$ is an $m$-step function on the unit interval whose $i$-th step is $r_i$. Note that

$$\begin{aligned} ||u||^2 &= \int_0^1 [u(t)]^2 \, dt = \sum_{i=1}^m \int_{(i-1)/m}^{i/m} [u(t)]^2 \, dt \\ &= \sum_{i=1}^m \int_{(i-1)/m}^{i/m} r_i^2 \, dt = (1/m) \sum_{i=1}^m r_i^2 = (1/m)||r||^2, \end{aligned}$$

so $||u|| = (1/\sqrt{m})||r||$ where $||r||$ is the Euclidean norm.

Let $\mathcal{B}(\mathbb{R}^m)$ be the linear operators on $\mathbb{R}^m$ with the operator norm

$$||A|| = \sup\{||Ar||/||r|| : \ r \in \mathbb{R}^m, r \neq 0\},$$

for any $A \in \mathcal{B}(\mathbb{R}^m)$. We identify $\mathcal{B}(\mathbb{R}^m)$ with the space of $m \times m$ real matrices, using the standard basis.

Let $\mathcal{B}(L^2[0,1])$ be the bounded linear operators on $L^2[0,1]$, writing $||\cdot||$ for the operator norm

$$||T|| = \sup\{||Tu||/||u|| : \ u \in L^2[0,1], u \neq 0\},$$

$T \in \mathcal{B}(L^2[0,1])$.

We will be concerned with the subspace of $\mathcal{B}(L^2[0,1])$ consisting of integral operators derived from $L^2$ kernels. If $\kappa \in L^2[0,1]^2$ is a square-integrable function on the unit square with norm

$$||\kappa|| = (\int_0^1 \int_0^1 |\kappa(s,t)|^2 \, ds \, dt)^{1/2},$$

we call it an $L^2$ kernel.

There is an integral operator $K \in \mathcal{B}(L^2[0,1])$ naturally asociated to $\kappa$ via

$$(Ku)(s) = \int_0^1 \kappa(s,t)u(t) \, dt \text{ for } u \in L^2[0,1], \; s \in [0,1].$$

The norm of the operator is bounded by that of the kernel:

$$||K|| \leq ||\kappa||.$$

If $A \in \mathcal{B}(\mathbb{R}^m)$, define the kernel $\kappa_A : [0,1]^2 \to \mathbb{R}$ by

$$\kappa_A(s,t) = ma_{ij}$$

if $(i-1)/m \leq s < i/m$ and $(j-1)/m \leq t < j/m$, where $A = (a_{ij})$ relative to the standard basis.

For a 0-1 $m \times m$ matrix $A$, $\kappa$ is the characteristic function of the region in the unit square corresponding to the nonzero entries of $A$, scaled by a factor of $m$. This construction is closely related to the notion of the inflation of a matrix (Halmos and Sunder 1978).

We now put the preceding steps together to obtain an embedding of $\mathcal{B}(\mathbb{R}^m)$ into $\mathcal{B}(L^2[0,1])$.

The embedding $\bar{J}_m : \mathcal{B}(\mathbb{R}^m) \longrightarrow \mathcal{B}(L^2[0,1])$ is defined by $\bar{J}_m(A) = \hat{A}$, where $A = (a_{ij})$ and

$$(\hat{A}u)(s) = \int_0^1 \kappa_A(s,t)u(t)\,dt \text{ for } u \in L^2[0,1].$$

So $\hat{A}$ is the bounded linear operator associated to the kernel derived from $A$.

Then $\hat{A}J_m = J_m A$, and $||\hat{A}|| = ||A||$, i.e. the embedding $\bar{J}_m$ is an isometry.

If $A \in \mathcal{B}(\mathbb{R}^m)$, we denote the spectral radius of $A$ by $\rho(A)$. If $A$ is symmetric, then $\rho(A) = ||A||$.

Similarly, if $T \in \mathcal{B}(L^2[0,1])$ is a bounded linear operator we write its spectral radius as $\rho(T)$, and note that if $T$ is self-adjoint then $\rho(T) = ||T||$.

Define the **symmetric core** $\underline{A}$ of a 0-1 square matrix $A = (a_{ij})$ by $\underline{A}_{ij} = \min(a_{ij}, a_{ji})$, so $\underline{A}$ is symmetric and $\underline{A}_{ij} \leq A_{ij}$.

Similarly, the **symmetric envelope** $\overline{A}$ of $A$, defined by $\overline{A}_{ij} = \max(a_{ij}, a_{ji})$, is symmetric and $\overline{A}_{ij} \geq A_{ij}$.

Then the spectral radii satisfy

$$\rho(\underline{A}) \leq \rho(A) \leq \rho(\overline{A}).$$

Define the kernel $\delta : [0,1]^2 \to \mathbb{R}$ by

$$\delta(s,t) = \begin{cases} 1 & \text{if } |s - 1/2| + |t - 1/2| \leq 1/2 \\ 0 & \text{otherwise} \end{cases}$$

$\delta$ is the characteristic function of a diamond-shaped region $\Diamond$ in the square. Clearly $\delta \in L^2[0,1]^2$. Let $D \in \mathcal{B}(L^2[0,1])$ be the associated kernel operator.

Misiurewicz and Nitecki showed that

$$\rho(D) = 2/\pi$$

by checking that $u(t) = \sin \pi t$ is a positive eigenfunction for $D$ with eigenvalue $2/\pi$.

Define the **deficiency** $d = d(A)$ of a 0-1 $m \times m$ matrix $A = (a_{ij})$ as $m^2$ times the area inside the central diamond $\Diamond \subset [0,1]^2$ where the kernel $\alpha$ of the operator $\hat{A}$ vanishes.

$d(A)$ can be thought of as the (fractional) number of zero entries of $A$ inside its 'central diamond'. For example, the deficiency of the zero matrix is $m^2/2$.

The **excess** $e = e(A)$ is similarly defined as $m^2$ times the area outside the central diamond $\Diamond \subset [0,1]^2$ where the kernel $\alpha$ of $\hat{A}$ is nonzero.

## Proposition

*If $A$ is a symmetric 0-1 $m \times m$ matrix with deficiency $d$ and excess $e$ then*

$$\rho(D) - \frac{\sqrt{d}}{m} \leq \frac{1}{m}\rho(A) \leq \rho(D) + \frac{\sqrt{e}}{m}.$$

*Since $\rho(D) = 2/\pi$, we have*

$$\frac{2}{\pi} - \frac{\sqrt{d}}{m} \leq \frac{1}{m}\rho(A) \leq \frac{2}{\pi} + \frac{\sqrt{e}}{m}.$$

The **Aztec diamond** $Z_{2k}$ has $2j$ ones centered in columns $j$ and $2k+1-j$ for $i \leq k$. (It isn't the induced matrix of a permutation.)

$$Z_{10} = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \end{bmatrix}$$

The Aztec diamond has excess $e(Z_{2k}) = 2k$. These matrices are the key to all the upper bounds, together with a Perron-Frobenius argument that no matrix satisfying the bound on column sums can have larger spectral radius.

The odd cycle $\theta_{11} = (1\ 6\ 11\ 5\ 2\ 8\ 9\ 3\ 4\ 10\ 7)$ has induced matrix

$$M(\theta_{11}) = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \end{bmatrix}$$

For $n$ odd, the induced matrix of the Misiurewicz-Nitecki cycle $\theta_n$ has deficiency $(n-1)/4$. This is used for the lower bounds for cycles.

For $n = 4l$, the maximum entropy permutation $\theta_n$ has symmetric matrix $M(\theta_n)$.

$$M(\theta_{12}) = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

For $n = 4l$ , the induced matrix of the maximum entropy permutation $\theta_n$ has deficiency $(n-2)/8$. This is used for the lower bounds for $4l$-permutations.

For $n = 4l + 2$, the maximum entropy permutation $\theta_n$ has matrix $M(\theta_n)$ which is not symmetric.

$$M(\theta_{10}) = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \end{bmatrix}$$

We take its symmetric core.

$$\underline{M}(\theta_{10}) = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \end{bmatrix}$$

For $n = 4l + 2$, the symmetric core $\underline{M}(\theta_n)$ has deficiency $(n-2)/4$.
This is used for the lower bound on such $n$-permutations.