Euclidean tilings

Invariant measures

Asymptotic Thurston norm

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Tilings of $\mathbb{R}^2$

Prototiles: $\mathcal{P} = \{p_1, \cdots, p_n\}$ is a finite set of polygons with colored edges.

Definition

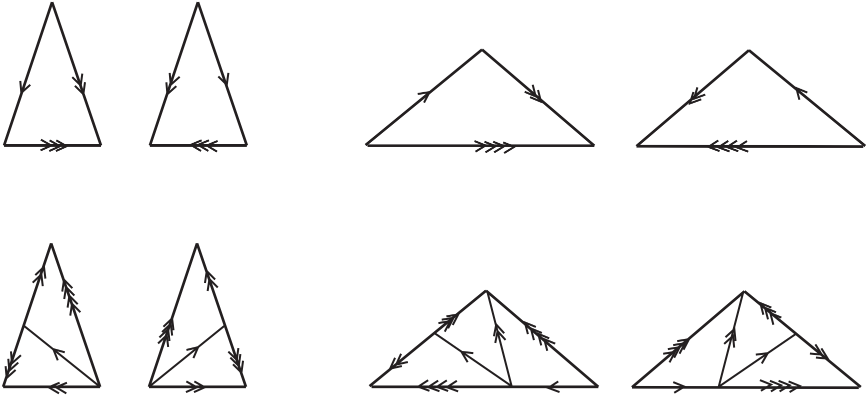
A $\mathcal{P}$-tiling of $\mathbb{R}^2$ is a collection of polygons with colored edges (t_i) (tiles) such that:

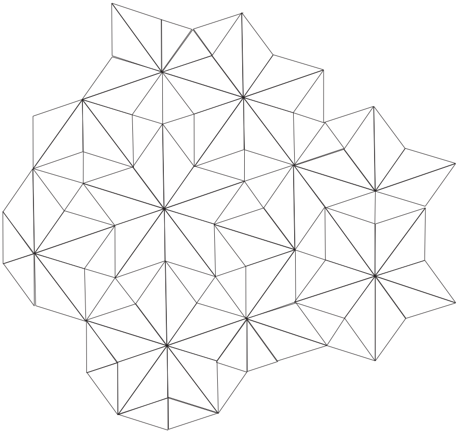
1. $\mathbb{R}^2 = \bigcup_i t_i$.
2. *The tiles t_i have disjoint interiors.*
3. *If two tiles t_i, t_j meet, they meet along edges whose colors match.*
4. *Each tile t_i is a translate of some prototile $p_j \in \mathcal{P}$.*

$\Omega_{\mathcal{P}}$ is the set of all $\mathcal{P}$-tilings.

Remark

$\Omega_{\mathcal{P}}$ might be empty: this is an undecidable problem.





The Anderson-Putnam complex $\mathcal{A}_{\mathcal{P}}$

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1. *The 2-cells are the prototiles p_j.*
2. *Two 2-cells are glued along the edges e_i, e_j if and only if there is a translation which carries e_i to e_j and the colors match.*

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2. *Two 2-cells are glued along the edges e_i, e_j if and only if there is a translation which carries e_i to e_j and the colors match. Orient the 2-cells with the orientation of the plane and choose an orientation for the edges.*
3. *Each edge has two sides: the collection of 2-cells where it appears with a $+$ sign in the boundary and the collection of 2-cells where it appears with a $-$ sign.*
   $\Rightarrow$ *Structure of Branched Surface*

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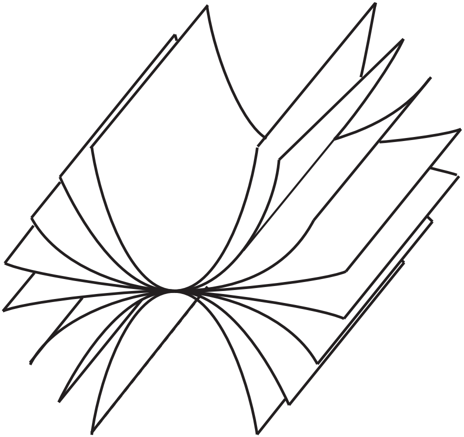

Homology and surfaces

$$H_2(\mathcal{A}_{\mathcal{P}};\mathbb{R}) = Ker(\partial\colon C_2(\mathcal{A}_{\mathcal{P}};\mathbb{R}) \to C_1(\mathcal{A}_{\mathcal{P}};\mathbb{R})).$$

Equivalent to look at the switch equations.

Lemma

Any non-negative integer 2-cycle $c \in H_2^+(\mathcal{A}_{\mathcal{P}};\mathbb{Z})$ is represented by a closed (i.e. with no boundary) compact surface S, denoted by $[S] = c$.

This surface S is not necessarily unique up to homeomorphism.

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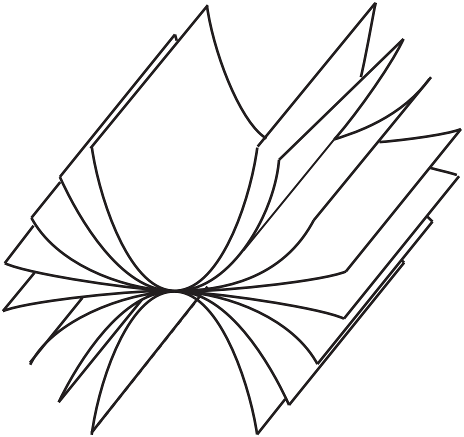

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Lemma

1. $||c|| = 0$ *if and only if there is a torus representing* c.
2. $||c_1 + c_2|| \leq ||c_1|| + ||c_2||$.
3. $||nc|| \leq |n| ||c||$.

It might happen $||nc|| < |n| ||c||$.

Asymptotic Thurston norm

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Remark

$|||c||| = 0$ does not imply that there is a torus representing c.

A geometric interpretation of the tiling problem

Theorem (Chazottes-Gambaudo-G)

$\Omega_{\mathcal{P}}$ is non-empty (which is equivalent to $\mathcal{P}$ tiles the plane) if and only if the asymptotic Thurston norm vanishes on some non-trivial class $c \in H_2^+(\mathcal{A}_{\mathcal{P}}; \mathbb{R})$.

Metrizable topology on $\Omega_{\mathcal{P}}$

$T, T' \in \Omega_{\mathcal{P}}$. $B_\epsilon(0)$: open ball of radius ϵ around the origin.

$A = \{\epsilon \in (0,1)$ s.t. there exists $u \in \mathbb{R}^2$ with $||u|| < \epsilon$ and

$$(T + u) \cap B_{1/\epsilon}(0) = T' \cap B_{1/\epsilon}(0)\}$$

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$\mathbb{R}^2$ amenable $\Rightarrow$ Existence of an invariant measure $\Rightarrow$ Existence of a non-negative real 2-cycle in $H_2^+(\mathcal{A}_{\mathcal{P}}; \mathbb{R})$.

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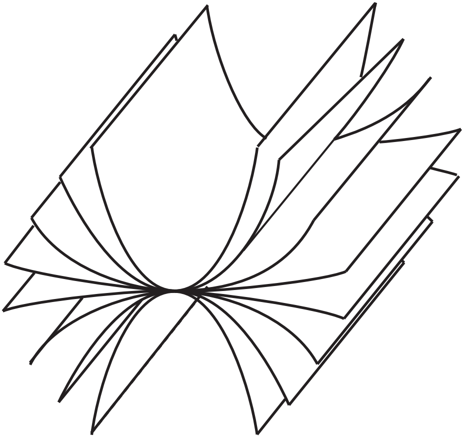

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Asymptotic Thurston norm and invariant measures

$\mathcal{M}(\Omega_{\mathcal{P}})$ set of invariant measures on $\Omega_{\mathcal{P}}$.

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Theorem (Chazottes-Gambaudo-G)

Let $c \in H_2^+(\mathcal{A}_{\mathcal{P}}; \mathbb{R})$. There exists $\mu \in \mathcal{M}(\Omega_{\mathcal{P}})$ such that $c = \pi(\mu)$ if and only if the asymptotic Thurston norm of c vanishes.

Wang tilings

A Wang tiling is (a tiling made from) a finite collection of unit squares with sides parallel to the axis of $\mathbb{R}^2$ and colored edges.

Theorem (Sadun-Williams)

For any finite collection of polygons $\mathcal{P}$ there is a Wang tiling $\mathcal{W}$ such that $(\Omega_{\mathcal{P}}, \mathbb{R}^2)$ and $(\Omega_{\mathcal{W}}, \mathbb{R}^2)$ are topologically equivalent.

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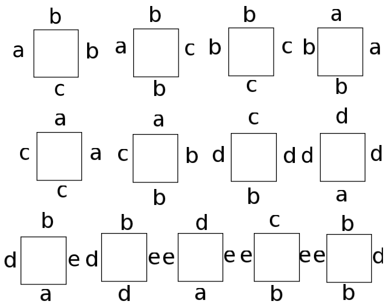

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Proposition

It is sufficient to prove our theorem for Wang tilings.

Hint of proof for Wang tilings

$c = \pi(\mu) \Rightarrow |||c||| = 0$: Forget the colors to obtain a new Wang tiling $\widehat{\mathcal{W}}$ and a new Anderson-Putnam complex $\mathcal{A}_{\widehat{\mathcal{W}}}$. The system $(\Omega_{\mathcal{W}}, \mathbb{R}^2)$ is a sub-system of $(\Omega_{\widehat{\mathcal{W}}}, \mathbb{R}^2)$.

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Periodic orbits of $\mathbb{R}^2$ (tori) are dense in $(\Omega_{\widehat{\mathcal{W}}}, \mathbb{R}^2)$.
Any invariant measure in $\mathcal{M}(\Omega_{\mathcal{W}})$ is an invariant measure in $\mathcal{M}(\Omega_{\widehat{\mathcal{W}}})$.

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$c = \pi(\mu)$ in $H_2^+(\mathcal{A}_{\mathcal{P}}; \mathbb{R})$ is approximated by a sequence of 2-cycles (c_i) in $H_2^+(\mathcal{A}_{\widehat{\mathcal{W}}}; \mathbb{R})$ such that $|||c_i||| = 0$.

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By continuity $|||c||| = 0$.

The reverse implication

$|||c||| = 0 \Rightarrow c = \pi(\mu)$: Inverse limit of branched surfaces $\mathcal{A}^p_{\mathcal{W}}$ which approximate $\Omega_{\mathcal{W}}$ made by squares of area p^2.
Let (c_i) sequence of rational cycles c_i with $\lim_{i\to\infty} c_i = c$. By continuity $|||c_i||| \to 0 \Rightarrow$ existence of surfaces S_i and integers n_i s.t. $\lim_{i\to\infty} \dfrac{|\chi(S_i)|}{n_i} = 0$. Fix $p > 0$ and consider for each i the subsurface (with boundary) formed by squares of area p^2 with associated chain $c_{i,p}$. Non-empty for i sufficiently large.

$$|c_{i,p} - n_i c_i| \leq K\chi(S_i)p^2$$

$c_{i,p}$ image of a chain $c^p_{i,p}$ for $\mathcal{A}^p_{\mathcal{W}}$. Let c^p normalized accumulation point. Since

$$|\partial c^p_{i,p}| \leq K\chi(S_i)p^2$$

c^p is a 2-cycle for $\mathcal{A}^p_{\mathcal{W}}$ which projects to c.
True for any $p > 0$. Since $\bigcap H_2^+(\mathcal{A}^p_{\mathcal{W}}, \mathbb{R}) = \mathcal{M}(\Omega_{\mathcal{W}})$, $c = \pi(\mu)$.