Minimal models for actions of amenable groups

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Definition

Dynamical systems (X, T) and (Y, S) are *Borel* isomorphic* if there exists an equivariant Borel-measurable bijection $\Phi : \tilde{X} \to \tilde{Y}$ between full invariant subsets $\tilde{X} \subset X$ and $\tilde{Y} \subset Y$, such that the conjugate map $\Phi^* : \mathcal{P}_T(X) \to \mathcal{P}_S(Y)$, $\Phi^*(\mu) = \mu \circ \Phi^{-1}$, is a (affine) homeomorphism with respect to weak* topologies.

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Theorem (Downarowicz, 2006)

If X is a metrizable, compact, zero-dimensional space and T has no periodic points then (X, T) is Borel isomorphic to a minimal topological dynamical system.*

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Theorem (F. and Huczek, 2014)

If X is a metrizable, compact, zero-dimensional space and an amenable group G acts freely on X
then (X, G) is Borel isomorphic to some minimal dynamical system (Y, G).*

G —a countable *amenable group*

There exists a sequence of finite sets $F_n \subset G$ (*Følner sets*) s.t.

$$\forall_{g \in G} \lim_{n \to \infty} \frac{|gF_n \triangle F_n|}{|F_n|} = 0,$$

where $gF = \{gf : f \in F\}$

1. $F_n \subset F_{n+1}$ for all n,
2. $e \in F_n$ for all n
3. $\bigcup_n F_n = G$,
4. $F_n = F_n^{-1}$.

$x \mapsto gx$ —a homeomorphism

The action of G is:

- *free* if $gx = x$ for any $g \in G$ and $x \in X$ implies that g is the neutral element
- *minimal* if for any $x \in X$ we have $\overline{\{gx : g \in G\}} = X$; equivalently, there are no non-trivial closed G-invariant subsets of X.

$\mathcal{P}_G(X)$ —set of all G-invariant Borel probability measures on X

Definition

The set $S \subset G$ is *syndetic* if there is a finite $F \subset G$ s.t. $SF = G$.

Definition

For $S \subset G$ and a finite $F \subset G$ define *lower Banach density* by

$$\begin{aligned} D_F(S) &= \inf_{g \in G} \frac{|S \cap Fg|}{|F|} \\ D(S) &= \sup\{D_F(S) \colon F \subset G, |F| < \infty\} \end{aligned}$$

Proposition

1. *If (F_n) is a Følner sequence then $D(S) = \lim_{n\to\infty} D_{F_n}(S)$.*
2. *S is syndetic if and only if $D(S) > 0$.*

$$\Lambda = (X \cup \{0, 1, *\})^{\mathbb{Z}}$$

$$d(x, y) = \begin{cases} d_X(x, y) & \text{for } x, y \in X \\ \mathrm{diam}(X) & \text{if } x \notin X \text{ or } y \notin X \end{cases}$$

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$$\mathbf{x} = \begin{bmatrix} \vdots \\ x^0 \\ x^1 \\ \vdots \end{bmatrix}, \mathbf{y} = \begin{bmatrix} \vdots \\ y^0 \\ y^1 \\ \vdots \end{bmatrix} \in \Lambda \implies d_\Lambda(\mathbf{x}, \mathbf{y}) = \sum_{i=-\infty}^{\infty} \frac{d(x^i, y^i)}{2^{|i|}}.$$

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Λ^G with $(gy)(h) = y(hg)$ —“a multidimensional shift space”

An array representation $\hat{X}$ of X is the range of a map $X \ni x \mapsto \hat{x} \in \Lambda^G$ defined by

$$\hat{x}(g)_n = \begin{cases} gx & \text{if } n = 0 \\ 0 & \text{otherwise} \end{cases}.$$

$\hat{X}$ is compact and G-invariant and $x \mapsto \hat{x}$ is a topological conjugacy.

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A block in Λ^G is a map $B\colon F \to \Lambda$, where F is finite.

A distance between blocks B_1, B_2 on a common domain F is

$$D(B_1, B_2) = \sup_{g \in F} d_\Lambda(B_1(g), B_2(g))$$

Lemma (Marker lemma)

For every finite $H \subset G$ there exists a clopen set V such that:

1. $g(V)$ *are disjoint for each* $g \in H$,
2. $\bigcup_{g \in F} g(V) = X$ *for some Følner set* F.

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Corollary (Hooks corollary)

For every $x \in X$ and a finite $T \subset G$ there is a set $C(x) \subset G$ s.t.

1. $Tg \cap Tg' = \emptyset$ *for each pair* $g, g' \in C(x)$, $g \neq g'$,
2. $(\exists F)(\forall x \in X)(\forall g \in G)(C(x) \cap Fg \neq \emptyset)$.

Also, $(\forall g \in G)(C(gx)g = C(x))$ *and* $x \mapsto C(x)$ *is continuous.*

Theorem (F. and Huczek, 2014)

If X is a metrizable, compact, zero-dimensional space and an amenable group G acts freely on X
then (X, G) is Borel isomorphic to some minimal dynamical system (Y, G).*

Lemma

Let:

1. $Y \subset \Lambda^G$ *be an array system*
2. $\mathcal{B}_Y$ *be the collection of all blocks occuring in* Y
3. $\mathcal{B}'_Y \subset \mathcal{B}_Y$ *be such that:*
   for every $\varepsilon > 0$ *and every* $B \in \mathcal{B}_Y$ *there exists* $B' \in \mathcal{B}'_Y$ *such that* $D(B, B'') < \varepsilon$ *for some subblock* B'' *of* B'.

If there exist a dense subset Y' *of* Y *consisting of elements* y *in which every* $B \in \mathcal{B}'_Y$ *occurs syndetically then the system* (Y, G) *is minimal.*

Step 1

1. $T_0 = \{e\}$
2. $\mathcal{B}_1 = (B^1_1, B^1_2, \dots, B^1_{N_1})$ —ε_1-dense set of blocks with domain T_0 occurring in $\hat{X}$ (ε_k—fixed summable sequence).
3. $C'_1(x)$ —a set of positive lower Banach density
4. F_{m_1} —a Følner set s.t. $F_{m_1}g$ contains at least N_1 elements $c \in C'_1(x)$ for each g and $\varepsilon_1 |F_{m_1}| > N_1$
5. $C_1(x)$ —a set of "hooks" for copies of F_{m_1}

6. $(d_1^c, \dots, d_{N_1}^c)$ —a subset of $F_{m_1} c \cap C_1'(x)$

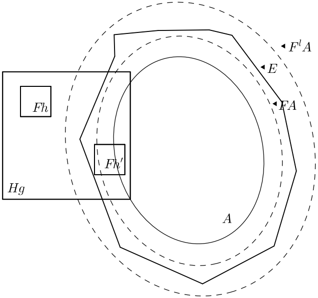
$$\Phi_1(x)_{n,d_j^c} = \begin{cases} * & \text{if } n = -1 \\ B_j^1(0) & \text{if } n = 0 \\ x_{0,d_j^c} & \text{if } n = 1 \end{cases}$$

7. $X_1 = \Phi_1(\hat{X})$

8. There exists a set $E \subset G$ such that $Eg \cap C_1(x)$ is nonempty for every $g \in G$.
Put $T_1 = F_{m_1} E$. Every block from $\mathcal{B}_1$ occurs in $\Phi_1(x)$ inside $T_1 g$ for each g.

Lemma (Window lemma)

Let $e \in F \subset G$, $l \in \mathbb{N}$. There is $H \supset F$ s.t. if $A \subset G$, $g \in G$ and $FA \subset E \subset F^l A$ then $Fh \subset Hg \cap E$ or $Fh \subset Hg \setminus E$ for some h.



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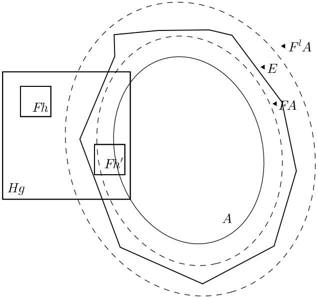

9. Choose H_1 using window lemma for $F = T_1$. The set H_1 will replace T_1 in the role of being a "syndeticity constant" for occurrence of elements of $\mathcal{B}_1$.

Step 2

1. $\mathcal{B}_2 = (B_1^2, B_2^2, \ldots, B_{N_2}^2)$ —ε_2-dense set of blocks with domain T_1 occurring in X_1
2. $C_2'(x)$ —obtained from "hooks" corollary for the set $H_1^{-1}T_1$. For distinct $c, c' \in C_2'(x)$ each set $H_1 g$ ($g \in G$) intersects at most one of $T_1 c$, $T_1 c'$
3. F_{m_2} —a Følner set s.t. $F_{m_2} g$ contains at least N_2 elements $c \in C_2'(x)$ such that $T_1 c \subset F_{m_2} g$ and $N_2 \left|T_1\right|^2 < \varepsilon_2 \left|F_{m_2}\right|$.
4. $C_2(x)$ —obtained from "hooks" corollary for the set F_{m_2}

5. $(d_1^c, \ldots, d_{N_2}^c) \subset F_{m_2} c \cap C_2'(x)$, $D_j(x) = \left\{ d_j^c : c \in C_2(x) \right\}$

$$\Psi_2(x)_{n,g} = \begin{cases} * & \text{if } n = -2 \text{ and } g \in D_j(x) \\ 1 & \text{for } n = -2, g \in T_1 d, d \in D_j(x), \\ & \quad \text{but } g \notin D_j(x), \\ x_{N,g} & \text{for } n = 2,\ g \in T_1 d, d \in D_j(x), \\ & \quad \text{where } N = \max\{i \colon x_{i,g} \neq 0\} \\ B_j^2(gd^{-1})(n) & \text{for } n = -1, 0, 1 \text{ and } g \text{ as above} \end{cases}$$

$$\Phi_2 = \Psi_2 \circ \Phi_1, \quad X_2 = \Psi_2(X_1) = \Phi_2(X)$$

6. Syndetic occurrence of $\mathcal{B}_2$:
There is a set $E \subset G$ such that $Eg \cap C_2(x)$ is nonempty for every $g \in G$.
Put $T_2 = F_{m_2}E$—then $T_2 g$ contains some $F_{m_2}c$, $c \in C_2(x)$, for every $g \in G$.

6. Syndetic occurrence of $\mathcal{B}_2$:
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7. Syndetic occurrence of $\mathcal{B}_1$:
   By the choice of H_1 there exists some h such that T_1h is a subset of H_1g that is either disjoint from all T_1c or is a subset of $T_1C_2'(x)$.
   But the set H_1g ($g \in G$) intersects at most one T_1c for $c \in C_2'(x)$, so in the second case it must be one of T_1c.
   Anyway, the block $x(T_1h)$ is a block occurring in X_1.

6. Syndetic occurrence of $\mathcal{B}_2$:
   There is a set $E \subset G$ such that $Eg \cap C_2(x)$ is nonempty for every $g \in G$.
   Put $T_2 = F_{m_2}E$—then $T_2 g$ contains some $F_{m_2}c$, $c \in C_2(x)$, for every $g \in G$.

7. Syndetic occurrence of $\mathcal{B}_1$:
   By the choice of H_1 there exists some h such that $T_1 h$ is a subset of $H_1 g$ that is either disjoint from all $T_1 c$ or is a subset of $T_1 C_2'(x)$.
   But the set $H_1 g$ ($g \in G$) intersects at most one $T_1 c$ for $c \in C_2'(x)$, so in the second case it must be one of $T_1 c$.
   Anyway, the block $x(T_1 h)$ is a block occurring in X_1.

8. Choose H_2 using window lemma for an appropriate F.

$\tilde{X}_k^e$ —a subset of X on which $\Phi_{k+1}(x)_e \neq \Phi_k(x)_e$.
$\mu(\tilde{X}_k^e) = O(\varepsilon_{k+1})$ for each G-invariant ergodic measure μ.
Therefore, $\tilde{X} = X \setminus \bigcup_{g\in G} \bigcap_{k=1}^{\infty} \bigcup_{j=k}^{\infty} g(\tilde{X}_j^e)$ is a full set.

$$\Phi(x) = \lim_{k\to\infty} \Phi_k(x)$$

Proposition

Let $Y = \overline{\Phi(\tilde{X})}$ in Λ^G.

1. *$\Phi(\tilde{X})$ is a full subset of Y.*
2. *Φ is an equivariant Borel-measurable bijection onto a full set.*
3. *Φ^* is an affine homeomorphism between simplices of invariant measures on X and Y.*

Thank you for your attention!