New approach to entropy

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The *Kolmogorov–Sinai entropy of T* is defined as

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where (F_n) is the *special* sequence of finite sets called a *Følner sequence*.

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Theorem 1 (B. Weiss)

If $\mathcal{G}$ is countable amenable, then

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Theorem 3 (D.)

If $\mathcal{G}$ is countable amenable, and $(X, \mathcal{U})$ is a *clopen partition* (hence a disjoint open cover), then

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or *monotonicity wrt. condition*

$$G \subset H \implies \mathrm{H}(F|G) \geq \mathrm{H}(F|H).$$

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Nevertheless, Theorems 2 and 3 hold for topological entropy. The tool to prove them is the *variational principle*.

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(This is not so obvious when looking at Følner sets.)

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EXAMPLE: full $\{0, 1\}$-shift on the free group $\mathcal{F}_2(a, b)$,
$\mathcal{P} = \{[0], [1]\}$, $\mathcal{Q} = \mathcal{P}^{\{e,a,b\}}$.

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B. Weiss can prove the Bernoulli shifts postulate for *sofic groups*.
Otherwise the problem is open.

THANK YOU